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Volterra Filtering and Higher Order Whiteness

P. Bondon, P. L. Combettes, and B. Picinbono

***Abstract*—Some properties of Volterra filtering are established. Finite-order, finite-horizon Volterra filtering is investigated as well as its asymptotic properties. Next, the concepts of Volterra unpredictability and uninterpolability lead to generalizations of the notion of white noise to higher orders. These generalizations are introduced and relations are established between them.**

I. INTRODUCTION

The problem of minimum mean-square (MS) error filtering of a stationary random process $(X_n)_{n\in\mathbf{Z}}$ in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ can be stated as follows: for every integer n, find the projection of X_n onto the closed vector subspace $\mathcal{V}_n(K) = \mathcal{L}^2(\Omega, \mathcal{K}_n, \mathsf{P})$ of $\mathcal{K}_n$-measurable random variables (RV's), where $\mathcal{K}_n$ denotes the σ-algebra generated by $(X_{n-k})_{k\in K}$ ($K \subset \mathbf{Z}^*$). The solution to this problem is $\mathsf{E}(X_n \mid \mathcal{K}_n)$, which is usually difficult to determine. Hence, it is customary to restrict the search for a best approximation to smaller subspaces, most notably to $\mathcal{V}_n(1, K)$, the closure of all linear combinations of $(X_{n-k})_{k\in K}$. This linear filtering operation is optimal if $(X_n)_{n\in\mathbf{Z}}$ is spherically invariant (in particular, Gaussian) in the sense that $\mathsf{E}(X_n \mid \mathcal{K}_n) \in \mathcal{V}_n(1, K)$ for such processes [1]. If $(X_n)_{n\in\mathbf{Z}}$ is not spherically invariant, the MS error can be reduced with respect to the linear case by projecting onto a closed vector subspace $\mathcal{V}'_n(K)$ such that $\mathcal{V}_n(1, K) \subset \mathcal{V}'_n(K) \subset \mathcal{V}_n(K)$. The larger $\mathcal{V}'_n(K)$, the smaller the MS error; on the other hand, $\mathcal{V}'_n(K)$ should be compatible with the amount of probabilistic information on $(X_n)_{n\in\mathbf{Z}}$ needed to compute the corresponding projection. Such nonlinear filters have been studied in several places in the literature. For instance, in [7], $\mathcal{V}'_n(K)$ is the span of $(g(X_{n-k}))_{k\in K}$, where K is finite and g is a polynomial; in [8], $\mathcal{V}'_n(K)$ is the span of $(X_{n-k_1}, X_{n-k_1}X_{n-k_2})_{(k_1,k_2)\in K^2}$, where K is finite.

In this correspondence, we establish some properties of Volterra filtering, in which the best MS approximation of X_n is sought in the closure $\mathcal{V}_n(q, K)$ of the span of all products of q or less terms from $(X_{n-k})_{k\in K}$, K being an arbitrary subset of $\mathbf{Z}^*$.[1] We first investigate finite-order, finite-horizon Volterra filtering and then establish asymptotic properties as the order of the filter and the size of the observation record increase. Next, we study processes that are unpredictable or uninterpolable by a Volterra filter. This yields various higher order extensions of the usual notion of (second-order) white noise, which is characterized by its unpredictability and its uninterpolability by a linear filter. Examples are provided to show that the relations between these higher order whitenesses are not trivial.

II. PRELIMINARIES

A. Assumptions and Notations

The RV's are assumed real-valued and defined on a fixed probability space $(\Omega, \mathcal{A}, \mathsf{P})$. No distinction is made between P-almost surely equal RV's. The Hilbert space of square-integrable RV's is denoted by $\mathcal{L}^2 = \mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$. Its scalar product is defined by $\langle X \mid Y \rangle = \mathsf{E}XY$ and its norm by $\|X\| = \mathsf{E}^{1/2}|X|^2$. The projection of $X \in \mathcal{L}^2$ onto a closed vector subspace $\mathcal{V} \subset \mathcal{L}^2$ is the unique RV $\dot{X} \in \mathcal{V}$ such that $\|X - \dot{X}\| = d(X, \mathcal{V}) = \inf_{V\in\mathcal{V}} \|X - V\|$, where d is the underlying distance. If $\mathcal{S}$ is an arbitrary nonempty family of RV's in $\mathcal{L}^2$, the subspace of all (finite) linear combinations of RV's in $\mathcal{S}$ is denoted by $\mathrm{sp}\,\mathcal{S}$ and its closure by $\overline{\mathrm{sp}}\,\mathcal{S}$. $\mathbf{N}$ is the set of nonnegative integers, $\mathbf{N}^*$ the set of positive integers, and $\mathbf{Z}^*$ the set of nonzero integers.

B. Best Linear Approximation

Let $\emptyset \neq \mathcal{S} \subset \mathcal{L}^2$ and $\mathcal{V} = \overline{\mathrm{sp}}\,\mathcal{S}$. The best linear MS approximation of a RV X in $\mathcal{L}^2$ relative to $\mathcal{S}$ is its projection $\dot{X}$ onto $\mathcal{V}$. It satisfies

$$(\forall V \in \mathcal{V}) \quad \langle X - \dot{X} \mid V \rangle = 0 \tag{1}$$

and the associated MS approximation error is

$$\varepsilon^2 = d(X, \mathcal{V})^2 = \|X\|^2 - \|\dot{X}\|^2 = \|X\|^2 - \langle X \mid \dot{X} \rangle. \tag{2}$$

[1] Such expansions were first studied by Fréchet [5] and Volterra [13]. A general account of the theory and its applications can be found in [9].

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Lemma 1: Let $(\mathcal{S}_m)_{m\in\mathbb{N}}$ be an increasing sequence of nonempty subsets of $\mathcal{L}^2$ and let $\mathcal{S} = \bigcup_{m\in\mathbb{N}}\mathcal{S}_m$. Let $\dot{X}_m$ and $\dot{X}$ be the best linear MS approximations of an RV X in $\mathcal{L}^2$ relative to $\mathcal{S}_m$ and $\mathcal{S}$, respectively, and let $(\varepsilon_m^2)_{m\in\mathbb{N}}$ and ε^2 be the associated MS approximation errors. Then, as m increases indefinitely, $\dot{X}_m \overset{\text{m.s.}}{\to} \dot{X}$ and $\varepsilon_m^2 \downarrow \varepsilon^2$.

Proof: Let $(\overline{\mathcal{V}}_m)_{m\in\mathbb{N}}$ be an increasing sequence of closed subspaces in a Hilbert space Ξ with distance d, let $\mathcal{V} = \bigcup_{m\in\mathbb{N}}\overline{\mathcal{V}}_m$ be its limit, and let x_m be the projection of $x \in \Xi$ onto $\overline{\mathcal{V}}_m$. Then: 1) $(x_m)_{m\in\mathbb{N}}$ converges strongly to the projection of x onto $\overline{\mathcal{V}}$; 2) $(d(x,\overline{\mathcal{V}}_m))_{m\in\mathbb{N}}$ is decreasing and converges to $d(x,\overline{\mathcal{V}})$ [10]. Now, let $\Xi = \mathcal{L}^2$ and $(\forall m \in \mathbb{N})\quad \mathcal{V}_m = \operatorname{sp}\mathcal{S}_m$. It follows from 1) that the sequence $(\dot{X}_m)_{m\in\mathbb{N}}$ of projections of X onto $(\overline{\operatorname{sp}}\,\mathcal{S}_m)_{m\in\mathbb{N}}$ converges in the MS sense to the projection $\ddot{X}$ of X onto $\bigcup_{m\in\mathbb{N}}\overline{\operatorname{sp}}\,\mathcal{S}_m = \overline{\mathcal{V}}$ and from 2) that the sequence of MS errors $(\varepsilon_m^2)_{m\in\mathbb{N}} = (d(X,\overline{\operatorname{sp}}\,\mathcal{S}_m)^2)_{m\in\mathbb{N}}$ decreases monotonically to $\ddot{\varepsilon}^2 = d(X,\overline{\mathcal{V}})^2$. But we observe that $\overline{\mathcal{V}} = \overline{\bigcup_{m\in\mathbb{N}}\operatorname{sp}\mathcal{S}_m}$. Moreover, it is clear that $\bigcup_{m\in\mathbb{N}}\operatorname{sp}\mathcal{S}_m \subset \operatorname{sp}\bigcup_{m\in\mathbb{N}}\mathcal{S}_m$. Conversely, since $(\mathcal{S}_m)_{m\in\mathbb{N}}$ is increasing, we have $\operatorname{sp}\bigcup_{m\in\mathbb{N}}\mathcal{S}_m \subset \bigcup_{m\in\mathbb{N}}\operatorname{sp}\mathcal{S}_m$. Hence, $\bigcup_{m\in\mathbb{N}}\operatorname{sp}\mathcal{S}_m = \operatorname{sp}\mathcal{S}$ and $\overline{\mathcal{V}} = \overline{\operatorname{sp}}\,\mathcal{S}$. Therefore, $\ddot{X} = \dot{X}$ and $\ddot{\varepsilon}^2 = \varepsilon^2$. ■

Now, suppose that $\mathcal{S}$ is a finite family, say $\mathcal{S} = (V_k)_{1\le k\le M}$. Then, we can write directly $\dot{X} = \sum_{k=1}^M h_k V_k = h^T V$, where $h = [h_1,\cdots,h_M]^T \in \mathbb{R}^M$ and $V = [V_1,\cdots,V_M]^T$. From (1), we get

$$(\forall l \in \{1,\cdots,M\})\ \sum_{k=1}^{M} h_k\langle V_k \mid V_l\rangle = \langle X \mid V_l\rangle \tag{3}$$

while (2) yields

$$\varepsilon^2 = \|X\|^2 - \sum_{k=1}^{M} h_k\langle X \mid V_k\rangle. \tag{4}$$

Define $\gamma = \mathbf{E}XV$ and $\Gamma = \mathbf{E}VV^T$. Then, (3) and (4) can be combined into

$$\begin{pmatrix}\varepsilon^2\\ 0\end{pmatrix} = \begin{pmatrix}\mathbf{E}X^2 & \gamma^T\\ \gamma & \Gamma\end{pmatrix}\begin{pmatrix}1\\ -h\end{pmatrix}. \tag{5}$$

It should be noted that if Γ is singular, h is not unique. However, the existence and uniqueness of $\dot{X}$ are guaranteed by the projection theorem. If Γ is nonsingular, it follows from (5) that the MS error can be expressed as

$$\varepsilon^2 = \frac{\begin{vmatrix}\mathbf{E}X^2 & \gamma^T\\ \gamma & \Gamma\end{vmatrix}}{|\Gamma|}.$$

III. Volterra Filtering

A. Generalities

$(X_n)_{n\in\mathbb{Z}} \subset \mathcal{L}^2$ is a zero-mean random sequence. All the cross moments (cross-cumulants) invoked thereafter are assumed to be finite and shift-invariant. Consider the problem of finding, for every integer n, the best linear MS approximation of X_n relative to the family[2]

$$\mathcal{S}_n(q,K) = \Big(\Big(\prod_{j=1}^{i} X_{n-k_j} - \mathbf{E}\prod_{j=1}^{i} X_{n-k_j}\Big)_{(k_1,\cdots,k_i)\in K^i}\Big)_{1\le i\le q} \tag{6}$$

where $K \subset \mathbb{Z}^*$ and $q \in \mathbb{N}^*$. In general, this problem will be referred to as a qth-order Volterra interpolation problem. If $K \subset \mathbb{N}^*$, then one obtains a prediction problem. The best qth-order Volterra interpolation of X_n based on $(X_{n-k})_{k\in K}$ is its projection, $\dot{X}_n(q,K)$, onto $\mathcal{V}_n(q,K) = \overline{\operatorname{sp}}\,\mathcal{S}_n(q,K)$. In addition, the best infinite order Volterra interpolation of X_n based on $(X_{n-k})_{k\in K}$ is its projection, $\dot{X}_n(+\infty,K)$, onto $\mathcal{V}_n(+\infty,K) = \overline{\operatorname{sp}}\,\mathcal{S}_n(+\infty,K)$, where $\mathcal{S}_n(+\infty,K) = \bigcup_{q\in\mathbb{N}^*}\mathcal{S}_n(q,K)$.

[2] If $\dot{X}$ is an MS approximation of X, where $\mathbf{E}X = 0$, then $\|X - (\dot{X} - \mathbf{E}\dot{X})\|^2 = \|X-\dot{X}\|^2 - \mathbf{E}^2\dot{X} \le \|X-\dot{X}\|^2$. Hence, one should seek a zero-mean approximation. By construction, this is guaranteed here.

B. Finite Case

Let q be a fixed positive integer and suppose that the process $(X_n)_{n\in\mathbb{Z}}$ is observed over a finite horizon, yielding an observation record $(X_{n-k})_{k\in K}$, where $\operatorname{card}K = N$. In this case, the family $\mathcal{S}_n(q,K)$ is also finite.

Proposition 1: The cardinality of $\mathcal{S}_n(q,K)$ is $M = C_{N+q}^q - 1$.

Proof: Let $\gamma(q,N)$ be the number of N-tuples $(i_1,\cdots,i_N)$ in $\mathbb{N}^N$ such that $0 \le \sum_{k=1}^N i_k \le q$. Taking, respectively, $i_1 = 0$ and $i_1 > 0$, we see that the sequence $(\gamma(q,N))_{(q,N)\in\mathbb{N}^{*2}}$ satisfies the recursion $\gamma(q,N) = \gamma(q,N-1) + \gamma(q-1,N)$ and $\gamma(q,1) = q+1$, $\gamma(1,N) = N+1$. By invoking the formula $C_n^p = C_{n-1}^p + C_{n-1}^{p-1}$, it is easily verified that these relations are satisfied by the sequence $(C_{N+q}^q)_{(q,N)\in\mathbb{N}^{*2}}$. As the cardinality of $\mathcal{S}_n(q,K)$ is $M = \gamma(q,N) - 1$, the proof is complete. ■

The qth-order Volterra filtering procedure is described by (5). The matrix Γ contains all the cross moments of $(X_{n-k})_{k\in K}$ up to order $2q$. On account of the shift-invariance assumption, $\mathbf{E}X_n^2$, γ, and Γ do not depend on the integer n. It was seen in Section II-B that the filter is unique when Γ is invertible. As Γ is symmetric and nonnegative definite, it is noninvertible if and only if the elements of $\mathcal{S}_n(q,K)$ are linearly dependent. If $N = 1$, this means that, for every integer n, the RV X_n takes at most q values and that these values are independent of n. If $N > 1$, the invertibility of Γ is more difficult to characterize. Nevertheless, it is obvious that if the RV's $(X_n)_{n\in\mathbb{Z}}$ take at most q values, then Γ is noninvertible. One can show that the converse is generally false, except if the RV's $(X_{n-k})_{k\in K}$ satisfy some higher order uncorrelation properties.

Of course, the filtering operation based on (6) encompasses well known cases. For instance, the linear filter is obtained for $q = 1$ and it requires only the second-order cross moments of $(X_{n-k})_{k\in K}$. The linear-quadratic filter is obtained for $q = 2$; solving (5) then requires knowledge of the cross moments of $(X_{n-k})_{k\in K}$ up to the fourth order. If $N = 1$, a single sample, say X_{n-p}, of the process is observed, and we obtain a 1-D polynomial filter with $\Gamma = [\mu_{k+l} - \mu_k\mu_l]_{1\le k,l\le q}$, where $\mu_i = \mathbf{E}X_0^i$ and $\gamma = [\mathbf{E}X_pX_0^l]_{1\le l\le q}^T$.

C. Asymptotic Properties

The asymptotic properties of Volterra filtering are presented in the following propositions, in which $(K_p)_{p\in\mathbb{N}}$ is an arbitrary increasing sequence of nonempty subsets of $\mathbb{Z}^*$ and $K = \bigcup_{p\in\mathbb{N}}K_p$.

Proposition 2: $(\forall n \in \mathbb{Z})(\forall p \in \mathbb{N})\dot{X}_n(q,K_p) \overset{\text{m.s.}}{\to} \dot{X}_n(+\infty,K_p)$ and $\mathbf{E}|X_n - \dot{X}_n(q,K_p)|^2 \downarrow \mathbf{E}|X_n - \dot{X}_n(+\infty,K_p)|^2$ as $q \to +\infty$.

Proof: Apply Lemma 1 to $(\mathcal{S}_n(q,K_p))_{q\in\mathbb{N}^*}$, whose limit is $\mathcal{S}_n(+\infty,K_p)$. ■

Proposition 3: $(\forall n \in \mathbb{Z})(\forall q \in \mathbb{N}^*)\ \dot{X}_n(q,K_p) \overset{\text{m.s.}}{\to} \dot{X}_n(q,K)$ and $\mathbf{E}|X_n - \dot{X}_n(q,K_p)|^2 \downarrow \mathbf{E}|X_n - \dot{X}_n(q,K)|^2$ as $p \to +\infty$.

Proof: Apply Lemma 1 to $(\mathcal{S}_n(q,K_p))_{p\in\mathbb{N}}$, whose limit is $\bigcup_{p\in\mathbb{N}}\mathcal{S}_n(q,K_p) = \mathcal{S}_n(q,\bigcup_{p\in\mathbb{N}}K_p) = \mathcal{S}_n(q,K)$. ■

Proposition 4: $(\forall n \in \mathbb{Z})\ \dot{X}_n(q,K_p) \overset{\text{m.s.}}{\to} \dot{X}_n(+\infty,K)$ and $\mathbf{E}|X_n - \dot{X}_n(q,K_p)|^2 \downarrow \mathbf{E}|X_n - \dot{X}_n(+\infty,K)|^2$ as $p \to +\infty$ and $q \to +\infty$.

Proof: Apply Lemma 1 to $(\mathcal{S}_n(q,K_p))_{p\in\mathbb{N},q\in\mathbb{N}^*}$, whose limit is $\bigcup_{p\in\mathbb{N}}\bigcup_{q\in\mathbb{N}^*}\mathcal{S}_n(q,K_p) = \bigcup_{q\in\mathbb{N}^*}\mathcal{S}_n(q,K) = \mathcal{S}_n(+\infty,K)$. ■

D. Unpredictability and Uninterpolability

In this section, we turn our attention to Volterra prediction and interpolation based on full observation records. It is well known that a linearly unpredictable second-order stationary sequence is characterized by the constancy of its spectral density. One is then naturally led to ask whether this property can be generalized if Volterra filters are considered in place of linear ones. To investigate this question, let us define the subspaces

$$\mathcal{P}_{qn} = \overline{\text{sp}}\, \mathcal{S}_n(q, \mathbb{N}^*) \tag{7}$$

$$\mathcal{D}_{qn} = \overline{\text{sp}}\, \mathcal{S}_n(q, \mathbb{Z}^*). \tag{8}$$

In addition, let us denote the cross moments and cumulants by

$$m_i(k_1, \cdots, k_{i-1}) = \mathsf{E} X_n X_{n-k_1}, \cdots, X_{n-k_{i-1}}$$
$$c_i(k_1, \cdots, k_{i-1}) = \mathsf{C}(X_n, X_{n-k_1}, \cdots, X_{n-k_{i-1}})$$

where C stands for the cumulant function [11]. On account of the shift-invariance of the cross cumulants of $(X_n)_{n\in\mathbb{Z}}$, we have

$$\begin{aligned} c_i(k_1, \cdots, k_{i-1}) &= c_i(k_{\sigma(1)}, \cdots, k_{\sigma(i-1)}) \\ &= c_i(k_{\sigma(1)} - k_{\sigma(j)}, \cdots, -k_{\sigma(j)}, \cdots \\ &\quad \cdots, k_{\sigma(i-1)} - k_{\sigma(j)}) \end{aligned} \tag{9}$$

where σ is any permutation on $\{1, \cdots, i-1\}$. Then, c_i is completely determined in the space $\mathbb{Z}^{i-1}$ by its values in the domain Δ_{i-1} defined by [2]

$$\Delta_{i-1} = \{(k_1, \ldots, k_{i-1}) \in \mathbb{Z}^{i-1} \mid 0 \le k_{i-1} \le \cdots \le k_1\}.$$

Of course, the same symmetry relation holds for the function m_i. As $(X_n)_{n\in\mathbb{Z}}$ is a zero-mean process, the relations between moments and cumulants give $m_i = c_i$ for $i \in \{2, 3\}$ [11]. If c_i is absolutely summable, its Fourier transform is called the ith-order polyspectral density of $(X_n)_{n\in\mathbb{Z}}$ and is denoted by $\widehat{c}_i$. The nonredundant region of computation of $\widehat{c}_i$, which is analogous to Δ_{i-1} of c_i, can be derived using (9) and is given in [3]. Linear unpredictability is then equivalent to constancy of $\widehat{c}_2$ and is commonly referred to as second-order whiteness. The natural extension of the concept of whiteness to higher orders leads us to adopt the following definition.

Definition 1: $(X_n)_{n\in\mathbb{Z}}$ is a qth-order white noise (WNq) if the functions $(\widehat{c}_i)_{2\le i\le q}$ are constant.

Let us notice that in [4] the maximum flatness of a given function $\widehat{c}_i$ was seen to arise in connection with the polyspectral estimation of non-Gaussian autoregressive processes. With the notations (6) and (7) [(6) and (8)], the qth-order Volterra prediction [interpolation] of X_n with full observation record is its projection onto $\mathcal{P}_{qn}$ [onto $\mathcal{D}_{qn}$]. $(X_n)_{n\in\mathbb{Z}}$ is said to be unpredictable [uninterpolable] by a qth-order Volterra filter if[3]

$$(\forall n \in \mathbb{Z})\ \ X_n \perp \mathcal{P}_{qn}\ \ [X_n \perp \mathcal{D}_{qn}]. \tag{10}$$

In the propositions below, we establish links between higher order whiteness and (10). A characterization of second-order whiteness is given in Proposition 5. Proposition 6 concerns unpredictability, whereas Propositions 7 and 8 deal with uninterpolability. Finally, Proposition 9 provides a characterization of fourth-order whiteness.

Proposition 5: $(X_n)_{n\in\mathbb{Z}}$ is unpredictable by a first-order Volterra filter $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ is uninterpolable by a first-order Volterra filter $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ is WN2.

[3]In these definitions, since $(X_n)_{n\in\mathbb{Z}}$ has mean zero, we can replace the subspaces $\mathcal{P}_{qn}$ and $\mathcal{D}_{qn}$ by those obtained without centering the RV's in (6).

Proof: $(\forall n \in \mathbb{Z}) X_n \perp \mathcal{P}_{1n}[X_n \perp \mathcal{D}_{1n}] \Leftrightarrow (\forall k \in \mathbb{N}^*)[(\forall k \in \mathbb{Z}^*)]\ m_2(k) = 0$. As $m_2 = c_2$ and c_2 is even, the result follows at once. ■

Proposition 6: $(X_n)_{n\in\mathbb{Z}}$ is unpredictable by a second-order Volterra filter $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ is WN2 and $(\forall (k_1, k_2) \in \Delta_2, k_2 \neq 0)\ c_3(k_1, k_2) = 0$.

It follows from Proposition 5 and $m_3 = c_3$ that $(\forall n \in \mathbb{Z})\ X_n \perp \mathcal{P}_{2n} \Leftrightarrow (X_n)_{n\in\mathbb{Z}}$ is WN2 and $(\forall (k_1, k_2) \in \mathbb{N}^{*2})\ c_3(k_1, k_2) = 0$. As c_3 is characterized by its values on Δ_2, the proof is complete. ■

The above property of c_3 can be characterized in the frequency domain by $\widehat{c}_3(\nu_1, \nu_2) = \widehat{f}(\nu_1) + \widehat{f}(\nu_2) + \widehat{f}(-\nu_1 - \nu_2)$, where $\widehat{f}$ is the Fourier transform of the function $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(k) = 0$ if $k < 0$, $f(0) = c_3(0, 0)/3$ and $f(k) = c_3(k, 0)$ if $k > 0$.

Thus, Propositions 5 and 6 indicate that the equivalence between unpredictability by a qth-order Volterra filter and WN$(q + 1)$ whiteness is limited to $q = 1$. We now examine the equivalence between uninterpolability and polyspectral flatness.

Proposition 7: $(X_n)_{n\in\mathbb{Z}}$ is uninterpolable by a second-order Volterra filter $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ is WN3.

Proof: We deduce from Proposition 5 and $m_3 = c_3$ that $(\forall n \in \mathbb{Z})\ X_n \perp \mathcal{D}_{2n} \Leftrightarrow (X_n)_{n\in\mathbb{Z}}$ is WN2 and $(\forall (k_1, k_2) \in \mathbb{Z}^{*2})\ c_3(k_1, k_2) = 0$. But, according to (9), $(\forall k \in \mathbb{Z}^*)\ c_3(k, 0) = c_3(-k, -k)$ and the result follows at once. ■

Proposition 8: $(X_n)_{n\in\mathbb{Z}}$ is uninterpolable by a third-order Volterra filter $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ is WN3 and $(\forall (k_1, k_2, k_3) \in \Delta_3, k_1 \neq k_2, k_3 \neq 0)\ c_4(k_1, k_2, k_3) = 0$.

Proof: According to Proposition 7, $(\forall n \in \mathbb{Z})\ X_n \perp \mathcal{D}_{3n} \Leftrightarrow (X_n)_{n\in\mathbb{Z}}$ is WN3 and $(\forall (k_1, k_2, k_3) \in \mathbb{Z}^{*3})\ m_4(k_1, k_2, k_3) = 0$. From (9), we get that $(\forall (k, p) \in \mathbb{N}^{*2}, k > p)\ m_4(k, 0, 0) = m_4(-k, -k, -k)$ and $m_4(k, p, 0) = m_4(-p, k - p, -p)$. As $(X_n)_{n\in\mathbb{Z}}$ is a zero-mean process, we have $(\forall (k_1, k_2, k_3) \in \mathbb{Z}^3)\ c_4(k_1, k_2, k_3) = m_4(k_1, k_2, k_3) - m_2(k_1)m_2(k_2 - k_3) - m_2(k_2)m_2(k_1 - k_3) - m_2(k_3)m_2(k_1 - k_2)$. Then the result is easily derived. ■

In the frequency domain, the above property of c_4 can be characterized by $\widehat{c}_4(\nu_1, \nu_2, \nu_3) = \widehat{g}(\nu_1 + \nu_2) + \widehat{g}(\nu_2 + \nu_3) + \widehat{g}(\nu_1 + \nu_3)$, where $\widehat{g}$ is the Fourier transform of the function $g : \mathbb{Z} \to \mathbb{R}$ defined by $g(0) = c_4(0, 0, 0)/3$ and $g(k) = c_4(k, k, 0)$ if $k \neq 0$.

In view of Propositions 5, 7, and 8, we conclude that the equivalence between uninterpolability by a qth-order Volterra filter and WN$(q + 1)$ whiteness is limited to $q = 1, 2$.

Proposition 9: $(X_n)_{n\in\mathbb{Z}}$ is WN4 $\Leftrightarrow$ $(X_n)_{n\in\mathbb{Z}}$ and $(X_n^2)_{n\in\mathbb{Z}}$ are, respectively, uninterpolable by a third- and a first-order Volterra filter.

Proof: According to Proposition 8, $(X_n)_{n\in\mathbb{Z}}$is WN4 $\Leftrightarrow$ $(\forall n \in \mathbb{Z})\ X_n \perp \mathcal{D}_{3n}$and $(\forall k \in \mathbb{N}^*)\ c_4(k, k, 0) = 0$. As $c_4(k, k, 0) = \mathsf{C}(X_k^2, X_0^2)$, the latter requirement is equivalent to $(X_n^2)_{n\in\mathbb{Z}}$ being WN2. The result then follows from Proposition 5. ■

E. Relations Between Whitenesses

In this section, we introduce additional notions of higher order white noise and establish relations with those defined previously.

Definition 2: $(X_n)_{n\in\mathbb{Z}}$ is a pure white noise (PWN) if it consists of independent RV's.

The notion of semi-independence introduced in the following definitions is adapted from [12].

Definition 3: Let $\mathcal{G}_n$ denote the σ-algebra generated by $(X_{n-k})_{k\in\mathbb{Z}^*}$ and let $\mathcal{D}_n = \mathcal{L}^2(\Omega, \mathcal{G}_n, \mathsf{P})$. $(X_n)_{n\in\mathbb{Z}}$ is a semi-independent white noise (SIWN) if it is uninterpolable, that is, $(\forall n \in \mathbb{Z})\ \mathsf{E}(X_n \mid \mathcal{G}_n) = \mathsf{E}X_n = 0$.

Definition 4: Let $\mathcal{F}_n$ denote the σ-algebra generated by $(X_{n-k})_{k\in\mathbb{N}^*}$ and let $\mathcal{P}_n = \mathcal{L}^2(\Omega, \mathcal{F}_n, \mathsf{P})$. $(X_n)_{n\in\mathbb{Z}}$ is a causal

semi-independent white noise (CSIWN) if it is unpredictable, that is, $(\forall n \in \mathbb{Z})\ \mathbf{E}(X_n \mid \mathcal{F}_n) = \mathbf{E}X_n = 0$.

Definition 5: $(X_n)_{n\in\mathbb{Z}}$ is a white noise of degree q (WN^q) [causal white noise of degree q (CWN^q)] if $(\forall k \in \mathbb{Z}^*)$ [$(\forall k \in \mathbb{N}^*)$] $(\forall i \in \{2,\cdots,q\})\ \ m_i(k,\cdots,k) = 0$.

Notice that $(X_n)_{n\in\mathbb{Z}}$ is WN^q if and only if $(\forall n \in \mathbb{Z})$ $X_n \perp \overline{\text{sp}}\,(X^i_{n-k} - \mu_i)_{k\in\mathbb{Z}^*,1\le i\le q-1}$ and that $(X_n)_{n\in\mathbb{Z}}$ is CWN^q if and only if $(\forall n \in \mathbb{Z})\ \ X_n \perp \overline{\text{sp}}\,(X^i_{n-k} - \mu_i)_{k\in\mathbb{N}^*,1\le i\le q-1}$. The definitions of white noises lead to the following relationships.

$$\begin{array}{ccccc} & & \text{PWN} & \Rightarrow & \text{WN}q \\ & & \Downarrow & & \Downarrow \\ \text{WN3} \Leftarrow & & \text{SIWN} & \Rightarrow & \text{WN}^q \\ & & \Downarrow & & \Downarrow \\ & & \text{CSIWN} & \Rightarrow & \text{CWN}^q \Rightarrow \text{WN2} \end{array} \tag{11}$$

SIWN⇒WN3 because if $(X_n)_{n\in\mathbb{Z}}$ is SIWN, then $(\forall (k_1,k_2) \in \mathbb{Z}^{*2})\ \mathbf{E}(X_n \mid X_{n-k_1}, X_{n-k_2}) = 0$. This is also a consequence of Proposition 7 since $(\forall n \in \mathbb{Z})\ \ X_n \perp \mathcal{D}_n \Rightarrow (\forall n \in \mathbb{Z})\ \ X_n \perp \mathcal{D}_{2n}$. WN$q$ ⇒WN^q is proved via the method of calculation of cumulants developed in [6]. It implies that $(X_n)_{n\in\mathbb{Z}}$ is WN^q ⇔ $(\forall k \in \mathbb{Z}^*)(\forall i \in \{2,\cdots,q\})\ \ c_i(k,\cdots,k) = 0$. On the other hand, it results from Definitions 1 and 5 that properties WN2, WN^2, and CWN^2 are equivalent.

It is clear from (11) that a hierarchy exists among the various types of whiteness, from the strongest (PWN) to the weakest (WN2). Of course, even in the case of PWN whiteness, the probabilistic properties of $(X_n)_{n\in\mathbb{Z}}$ are not totally determined since the finite dimensional distribution functions are not specified. By introducing certain assumptions on the distribution functions, one may establish additional relationships between the various types of white noises. Thus, it is well known that a Gaussian sequence that is WN2 is also PWN and, therefore, all types of higher order whiteness coincide in this case.

We shall now present some counterexamples to show that the relationships in (11) cannot be improved upon in general. Hereafter, $(U_n)_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed zero-mean RV's whose moments $(\mathbf{E}|U_0|^i)_{i\in\mathbb{N}^*}$ are finite.

Proposition 10: $(X_n)_{n\in\mathbb{Z}}$ can be SIWN without being WN4.

Proof: Let $(Z_n)_{n\in\mathbb{Z}} \subset \mathcal{L}^2$ be a sequence of independent and identically distributed zero-mean RV's that is independent from $(U_n)_{n\in\mathbb{Z}}$ and let $(\forall n \in \mathbb{Z})\ \ X_n = Z_n(U_n + U_{n-1})$. Then $\mathbf{E}(X_n \mid \mathcal{G}_n) = \mathbf{E}Z_n\mathbf{E}(U_n + U_{n-1} \mid \mathcal{G}_n) = 0$. Now, it is readily seen that $\mathbf{C}(X_n^2, X_{n-1}^2) = \mathbf{E}^2 Z_0^2(\mathbf{E}U_0^4 - \mathbf{E}^2U_0^2)$. Therefore, if $\mathbf{E}U_0^4 \neq \mathbf{E}^2U_0^2$, i.e., the probability distribution of U_0 is not concentrated on at most two points, then $(X_n^2)_{n\in\mathbb{Z}}$ is not WN2 and it follows from Propositions 5 and 9 that $(X_n)_{n\in\mathbb{Z}}$ is not WN4. ■

We deduce from Propositions 10 and 5 that $(X_n)_{n\in\mathbb{Z}}$ can be uninterpolable while $(X_n^2)_{n\in\mathbb{Z}}$ is not unpredictable by a first-order Volterra filter.

Proposition 11: $(X_n)_{n\in\mathbb{Z}}$ can be CSIWN without being WN3.

Proof: Let $(\forall n \in \mathbb{Z})\ \ X_n = U_{n-1}^2U_n$. Since $\mathbf{E}U_0 = 0$, we have $(\forall n \in \mathbb{Z})\ \mathbf{E}(X_n \mid \mathcal{F}_n) = 0$. On the other hand, $\mathbf{E}X_nX_{n+1}^2 = \mathbf{E}^2U_0^2\,\mathbf{E}U_0^5$. Therefore, if $\mathbf{E}U_0^5 \neq 0$, $(X_n)_{n\in\mathbb{Z}}$ is not WN3. ■

As a consequence of Propositions 11 and 7, $(X_n)_{n\in\mathbb{Z}}$ can be unpredictable while it is not uninterpolable by a second-order Volterra filter.

Proposition 12: $(X_n)_{n\in\mathbb{Z}}$ can be WN3 without being CWN^4.

Proof: Let $(\forall n \in \mathbb{Z})\ \ X_n = U_n^2U_{n-1}$. One verifies easily that $(X_n)_{n\in\mathbb{Z}}$ is WN3 ⇔ $\mathbf{E}U_0^5 = 0$. On the other hand, $m_4(1,1,1) = \mathbf{E}U_0^2\,\mathbf{E}U_0^3\,\mathbf{E}U_0^7$, which can be nonzero. ■

It results from Proposition 12 and (11) that $(X_n)_{n\in\mathbb{Z}}$ can be WN3 without being CSIWN.

Proposition 13: A WN2 process can be predictable.

Proof: Let $(\forall n \in \mathbb{N}^*)\ \ X_n = \cos(n\Phi)$, where Φ is uniformly distributed over $[0, 2\pi]$. Then, $(X_n)_{n\in\mathbb{N}^*}$ is stationary and WN2. Nonetheless, $(X_n)_{n\in\mathbb{N}^*}$ can be expressed in terms of the RV's $(X_{n-k})_{0<k<n}$ by trigonometric formulae. Hence, $(\forall n \in \mathbb{N}^*)\ \ X_n \in \mathcal{P}_n$. ■

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Estimating 2-D DOA Angles Using Nonlinear Array Configurations

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***Abstract*—This paper modifies the direction-of-arrival matrix (DOAM) method for detecting 2-D direction-of-arrival (DOA) angles of narrowband far-field source signals using nonlinear array configurations. The key modification to the DOAM method is the replacement of the 1-D data model with a 2-D data model. The modified DOA matrix (MDOAM) method requires less computation, detects more sources with the same number of sensors, and removes the constraint on the spacing between subarrays.**

I. Introduction

ESPRIT [1]–[3] is a pioneering subspace rotation technique for locating narrowband signal sources assumed to be in the far-field

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