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Best Stable and Invertible Approximations for ARMA Systems

Patrick L. Combettes and H. Joel Trussell

***Abstract*—A method is proposed for finding a best stable and invertible approximation for an autoregressive moving average (ARMA) system, relative to a general quadratic metric in the coefficient space. Mathematically, the problem is equivalent to projecting the regression and moving average vectors of the system onto the set $\mathbb{S}$ of coefficients of monic Schur polynomials. The geometry of $\mathbb{S}$ is too complex to allow the problem to be approached directly in the ARMA coefficient space. A solution is obtained by constrained steepest descent in the hypercube of reflection coefficients, which is homeomorphic to $\mathbb{S}$.**

I. Introduction

An autoregressive moving average (ARMA)(p, q) system can be represented by the difference equation

$$(\forall n \in \mathbb{Z})\ X_n - \sum_{i=1}^{p} a_i X_{n-i} = U_n - \sum_{i=1}^{q} c_i U_{n-i} \qquad (1)$$

where $(U_n)_{n\in\mathbb{Z}}$ and $(X_n)_{n\in\mathbb{Z}}$ denote the input and output processes, respectively. It will be assumed that the system is causal with known orders p and q and that the numerator and the denominator of the transfer function of the system have no common factors. Let $\mathbb{S}_r$ be the set of coefficients of rth order monic Schur polynomials, that is

$$\mathbb{S}_r = \{\boldsymbol{v} \in \mathbb{R}^r | (\forall z \in \mathbb{C})\ z^r - \sum_{i=1}^{r} v_i z^{r-i} = 0 \Rightarrow |z| < 1\}. \qquad (2)$$

The system is said to be stable if its regression vector $\boldsymbol{a} = [a_1 \cdots a_p]^T$ lies in $\mathbb{S}_p$ and invertible if its moving average vector $\boldsymbol{c} = [c_1 \cdots c_q]^T$ lies in $\mathbb{S}_q$.[1] In this correspondence we propose a method for finding a best stable and invertible approximation to an ARMA (p, q) system. In geometrical terms, a best stable approximation is obtained by projecting $\boldsymbol{a}$ onto $\mathbb{S}_p$ and a best invertible approximation is obtained by projecting $\boldsymbol{b}$ onto $\mathbb{S}_q$. The projection onto $\mathbb{S}_r$ will be determined in the Hilbert space $\mathbb{R}^r$ obtained by endowing the Cartesian space $\mathbb{R}^r$ with the distance

$$(\forall (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^r \times \mathbb{R}^r) \qquad d(\boldsymbol{u}, \boldsymbol{v}) = [(\boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{Q} (\boldsymbol{u} - \boldsymbol{v})]^{1/2} \qquad (3)$$

where $\boldsymbol{Q}$ is a symmetric, positive definite, $r \times r$ matrix.

There are two major motivations for this work. The first one is found in the problem of identifying ARMA systems via set theoretic estimation. In this context, a set theoretic estimate of the ARMA coefficients is any object $[a_1 \cdots a_p c_1 \cdots c_q]^T$ lying in the intersection of constraint sets. As shown in [6], such an estimate can be obtained as any cluster point of the sequence generated by projecting an initial point sequentially onto each constraint set in a cyclic fashion. In particular, a method for projecting onto $\mathbb{S}_p \times \mathbb{S}_q$ is needed to enforce the stability and invertibility constraints at each iteration. The second motivation stems from the fact that methods for synthesizing or estimating ARMA parameters do not always enforce system stability and invertibility. In such instances, one needs a rule for assigning a stable regression vector to an unstable one and an invertible moving average vector to a noninvertible one. A projection algorithm can be used to this end and has the property of enforcing stability and invertibility while causing minimum adjustment in the system coefficients, thus preserving the properties of the original system to some extent. As will be seen in Section IV, this is not true of common stabilization procedures, which generally produce significant perturbations in the system coefficients, thereby inducing undesirable effects.[2] The following discussion will address primarily the problem of projecting the regression vector $\boldsymbol{a}$ onto the stability set $\mathbb{S}_p$. It is understood that the problem of projecting a moving average vector $\boldsymbol{c}$ onto the invertibility set $\mathbb{S}_q$ is identical. As will be seen in Section II, the geometrical complexity of $\mathbb{S}_p$ makes it very difficult to determine projections via constrained minimization in the regression space for arbitrary p. In the method presented here, a projection onto $\mathbb{S}_p$ is obtained by direct steepest descent in the stability hypercube of the reflection space. A method for projecting onto $\mathbb{S}_p$ was developed independently in [12], in which a solution is estimated iteratively by alternating minimizations with respect to each of the p reflection coefficients at each iteration. This simple scheme is easy to implement but its convergence properties are known to be poorer than those of steepest descent [10].

II. The Geometry of Stability

In statistics and engineering, the stability of ARMA systems has mainly been approached analytically [1], [8], [16]. In this section, the focus is placed of the geometry of the stability set in the regression space and in the reflection space. The geometric properties will give added insight into the effects of various stabilization procedures.

The Levinson operator [9] $\mathfrak{L}$ is a p-linear transformation[3] from $\mathbb{R}^p$ into $\mathbb{R}^p$ which maps a reflection vector $\boldsymbol{k} = [k_1 \cdots k_p]^T$ into a regression vector $\boldsymbol{a} = [a_1 \cdots a_p]^T = \boldsymbol{a}^{(p)}$ according to the recur-

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[1] Invertibility implies minimum phase and guarantees that the causal inverse filter is stable. From the standpoint of time series analysis, if $(U_n)_{n\in\mathbb{Z}}$ is a white noise process in (1), $\boldsymbol{a}$ being in $\mathbb{S}_p$ is a criterion for the stationarity of the ARMA(p, q) process $(X_n)_{n\in\mathbb{Z}}$ while invertibility implies identifiability since it ensures that there is a unique moving average vector which corresponds to any given form of the correlation function or of the spectral density of $(X_n)_{n\in\mathbb{Z}}$ [5].

[2] For instance, consider linear prediction. Let $\boldsymbol{b}$ be the pth order optimal prediction vector and let $\boldsymbol{R}$ be the correlation matrix of the data process. Suppose that $\boldsymbol{b}$ is unstable and that $\boldsymbol{a}$ is a stabilized version. The excess residual power generated by the suboptimal filter $\boldsymbol{a}$ is given by $(\boldsymbol{a} - \boldsymbol{b})^T \boldsymbol{R} (\boldsymbol{a} - \boldsymbol{b})$ [3]. Thus, in order to stabilize $\boldsymbol{b}$ while minimizing the excess error, one should choose an $\boldsymbol{a}$ in $\mathbb{S}_p$ that minimizes $d(\boldsymbol{a}, \boldsymbol{b})$ in (3), where $\boldsymbol{Q} = \boldsymbol{R}$.

[3] A transformation defined on $\mathbb{R}^p$ is said to be p-linear if, when any $p - 1$ of its variables is held fixed, it is linear with respect to the pth.

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sion:

$$\begin{aligned}&\text{for } i = 1, \cdots, p\\&\quad a_i^{(i)} = k_i\\&\quad \text{for } j = 1, \cdots, i-1\\&\qquad a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}.\end{aligned}$$

The geometry of stability in the reflection space is quite simple since the stability set is the hypercube $\mathcal{K}_p =]-1, 1[^p$. In the regression space, the stability set is well known for low orders: $\mathcal{S}_1 =]-1, 1[$ and $\mathcal{S}_2$ is the interior of the triangle with vertices $[-2, -1]^T$, $[0, 1]^T$, and $[2, -1]^T$. For $p > 2$, $\mathcal{S}_p$ is a geometrically complex nonconvex set defined by sets of inequality constraints [1], [2], [4], [8], [15], [16]. The restriction of $\mathcal{L}$ to $\mathcal{K}_p$ is a bijection onto $\mathcal{S}_p$ which is continuously differentiable, and so is its inverse [2].[4] Moreover, $\mathcal{L}$ maps the boundary $\partial\mathcal{S}_p$ of $\mathcal{S}_p$ onto the boundary $\partial\mathcal{K}_p$ of $\mathcal{K}_p$, but does not admit an inverse on $\partial\mathcal{S}_p$ [13]. In addition, $\partial\mathcal{S}_p$ possesses $p + 1$ vertices which are the images under $\mathcal{L}$ of the 2^p vertices of $\mathcal{K}_p$ and are tabulated for various values of p in [15]. For instance, the set $\mathcal{S}_3$, which is shown in Fig. 1, has vertices $[-3, -3, -1]^T$, $[-1, 1, 1]^T$, $[1, 1, -1]^T$, and $[3, -3, 1]^T$. To understand the geometry of $\mathcal{S}_p$ more concretely, let us define

$$\mathcal{S}_p^- = \left\{\boldsymbol{a} \in \mathbb{R}^p \,\middle|\, \sum_{i=1}^{p} |a_i| < 1\right\} \quad \text{and}$$

$$\mathcal{S}_p^+ = \bigcap_{i=1}^{p} \{\boldsymbol{a} \in \mathbb{R}^p \,|\, |a_i| < C_p^i\} \tag{4}$$

where $C_p^i = p!/(i!(p-i)!)$. The condition defining $\mathcal{S}_p^-$ is sufficient for stability while that defining $\mathcal{S}_p^+$ is necessary [13], which implies $\mathcal{S}_p^- \subset \mathcal{S}_p \subset \mathcal{S}_p^+$. One can regard $\mathcal{S}_p$ as a star-shaped body with $p + 1$ legs and circumscribed by the hyperrectangle $\mathcal{S}_p^+$; each leg originates at a vertex of $\mathcal{S}_p$ and connects to the diamond $\mathcal{S}_p^-$. Moreover, since the vertices of $\mathcal{S}_p^-$ are fixed points of $\mathcal{L}$ and belong to $\partial\mathcal{K}_p$, they also belong to $\partial\mathcal{S}_p$. The simple stability tests defining $\mathcal{S}_p^-$ and $\mathcal{S}_p^+$ are sometimes used in lieu of the exact stability criterion, especially when dealing with adaptive systems. However, let us stress here that both $\mathcal{S}_p^-$ and $\mathcal{S}_p^+$ constitute very poor approximations of $\mathcal{S}_p$. To see this, let us compute the Lebesgue measure μ of these sets. It is easily established that $\mu(\mathcal{S}_p^-) = 2^p/p!$ and $\mu(\mathcal{S}_p^+) = 2^p \Pi_{i=1}^{p-1} C_p^i$. Moreover, for every positive integer p, the measure of the stability set can be obtained by the following recursion, where $\mu(\mathcal{S}_1) = 2$ [11]:

$$\mu(\mathcal{S}_{p+1}) = d_n \mu(\mathcal{S}_p) \qquad \text{where } n = \left\lceil \frac{p}{2} \right\rceil \quad \text{and } d_n = \frac{(n!)^2 2^{(2n+1)}}{(2n+1)!}. \tag{5}$$

Table I gives the measure of these sets up to order 6. Clearly, neither the sufficient nor the necessary stability sets should be used to enforce stability.

III. Projection Algorithm

Let $\boldsymbol{b}$ be an arbitrary regression vector. In this section, we develop an algorithm to compute a projection of $\boldsymbol{b}$ onto $\mathcal{S}_p$ in $\mathbb{R}^p$, i.e., a minimum, over $\mathcal{S}_p$, of the objective function

$$\Psi(\boldsymbol{a}) = \tfrac{1}{2}(\boldsymbol{a} - \boldsymbol{b})^T \boldsymbol{Q}(\boldsymbol{a} - \boldsymbol{b}). \tag{6}$$

[4]In topological terms, $\mathcal{S}_p$ and $\mathcal{K}_p$ are homeomorphic.

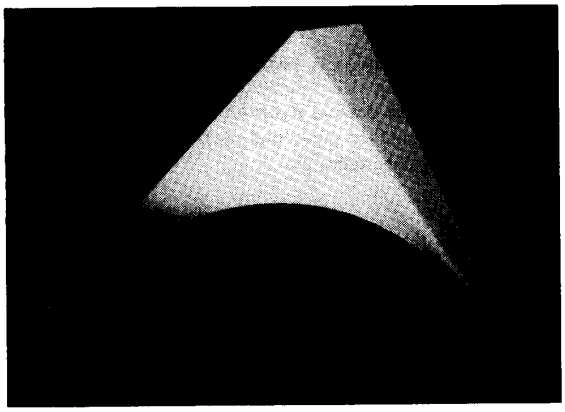


Fig. 1. The three-dimensional stability set.

TABLE I
Measure of Stability Sets

p	$\mu(\mathcal{S}_p^-)$	$\mu(\mathcal{S}_p)$	$\mu(\mathcal{S}_p^+)$
1	2.00	2.00	2.00
2	2.00	4.00	8.00
3	1.33	5.33	72.0
4	0.667	7.11	1536
5	0.267	7.59	80000
6	0.089	8.09	10368000

A. Existence and Uniqueness

Every point admits at least one projection onto a nonempty subset $\mathcal{S}$ of a finite dimensional normed vector space if and only if $\mathcal{S}$ is closed [6]. Since $\mathcal{S}_p$ is open, it is necessary to approximate it by some closed set. Note that every element in $\mathcal{S}_p$ can be approached within any degree of accuracy by an element of the closed set

$$\tilde{\mathcal{S}}_p = \mathcal{L}(\tilde{\mathcal{K}}_p) \qquad \text{with } \tilde{\mathcal{K}}_p = [\delta - 1, 1 - \delta]^p \tag{7}$$

where δ is a small positive number. Henceforth, for simplicity, we shall not distinguish $\tilde{\mathcal{S}}_p$ from $\mathcal{S}_p$ and $\tilde{\mathcal{K}}_p$ from $\mathcal{K}_p$.

Proposition: Let μ denote the Lebesgue measure in $\mathbb{R}^p$. Then μ-almost every point in $\mathbb{R}^p$ admits a unique projection onto $\mathcal{S}_p$.

Proof: The existence of a projection is guaranteed by the fact that $\mathcal{S}_p$ is nonempty and closed. The μ-almost everywhere uniqueness results from the fact that the subset of $\mathbb{R}^p$ of points with more than one projection onto a nonempty closed set has Lebesgue measure zero [7].[5]

The practical significance of this result is that almost every ARMA (p, q) system admits a unique best stable and invertible approximation.

B. The Algorithm

Performing the constrained minimization in the regression space for arbitrary p is difficult because of the geometrical complexity of $\mathcal{S}_p$. On the other hand, since the stability set in the reflection space is very simple, the minimization will be carried out there. For every $\boldsymbol{a}$ in $\mathcal{S}_p$, by invoking the Levinson transformation, the objective function displayed in (6) has a well-defined representation over $\mathcal{K}_p$ given by $\Phi(\boldsymbol{k}) = 0.5(\mathcal{L}(\boldsymbol{k}) - \boldsymbol{b})^T \boldsymbol{Q}(\mathcal{L}(\boldsymbol{k}) - \boldsymbol{b})$. Thus, a solution

[5]This remarkable result is widely unknown in approximation theory.

to the problem is a vector $\hat{\boldsymbol{a}} = \mathcal{L}(\hat{\boldsymbol{k}})$, where

$$\Phi(\hat{\boldsymbol{k}}) = \min \{\Phi(\boldsymbol{k}) | \boldsymbol{k} \in \mathcal{K}_p\}. \tag{8}$$

A constrained steepest descent algorithm can be used to solve this problem. Let $\mathcal{P}$ denote the projection operator onto $\mathcal{K}_p$ which enforces the stability constraint, i.e.,

$$(\forall \boldsymbol{k} \in \mathbb{R}^p)\ [\mathcal{P}(\boldsymbol{k})]_i = \begin{cases} 1 - \delta & \text{if } k_i \geq 1 - \delta \\ k_i & \text{if } |k_i| \leq 1 - \delta \\ \delta - 1 & \text{if } k_i \leq \delta - 1. \end{cases} \tag{9}$$

Given an initial vector $\boldsymbol{k}_0$ in $\mathcal{K}_p$ one performs the recursion

$$\boldsymbol{k}_{n+1} = \mathcal{P}(\boldsymbol{k}_n - \lambda_n \nabla \Phi(\boldsymbol{k}_n)) \tag{10}$$

where the positive constant λ_n is the step size at iteration n and where the gradient of Φ with respect to $\boldsymbol{k}$ is given by $\nabla \Phi(\boldsymbol{k}) = \boldsymbol{J}(\boldsymbol{k})\boldsymbol{Q}(\mathcal{L}(\boldsymbol{k}) - \boldsymbol{b})$ with $J_{ij}(\boldsymbol{k}) = \partial \mathcal{L}_j(\boldsymbol{k})/\partial k_i$. The algorithm is stopped at the smallest iteration index n such that

$$|\Phi(\boldsymbol{k}_n) - \Phi(\boldsymbol{k}_{n+1})| \leq \epsilon \tag{11}$$

where ϵ is some positive constant reflecting the degree of accuracy desired.

C. Convergence Properties

The convergence properties of the algorithm are conditioned by the step sizes $(\lambda_n)_{n \geq 0}$, the constant ϵ associated with the stopping rule (11), and the initial point $\boldsymbol{k}_0$. Ideally, the step sizes should be chosen so as to optimize the progression of the algorithm. Since optimal step sizes can never be evaluated exactly in practice, they will be selected by the standard adaptive procedure proposed in [14], which will be described in the next section. On the other hand, the determination of the constant ϵ should be consistent with the degree of accuracy required in a given application. It is worth noting that ϵ should not be made unreasonably small, for the iterations may start to oscillate indefinitely around the solution. Finally, since $\mathcal{S}_p$ is not convex, in choosing $\boldsymbol{k}_0$, one should be aware of the possibility of local convergence. Theoretically, the algorithm will converge to a local minimum of Φ and the solution it provides may be accepted as a projection only if it can be guaranteed to be global. In extensive simulation up to order $p = 8$, we were able to observe a lack of global convergence only in cases where $\mathcal{L}(\boldsymbol{k}_0)$ was near a vertex of $\mathcal{S}_p$ distant from the unstable point. This observation suggests that they are no "bumps" on $\partial \mathcal{S}_p$, which is not surprising if we go back to geometrical considerations. Indeed, due to the p-linearity of $\mathcal{L}$, the bounding hypersurfaces of $\mathcal{S}_p$ are generated by moving a straight line segment and at least one of them passes through all the vertices; moreover, vertices are connected by line segments in $\partial \mathcal{S}_p$ [15]. As a conclusion, we recommend that the iterations be started at whatever stable point is believed to be in some rough neighborhood of the true solution. For instance, if the unstable vector is slightly outside $\mathcal{S}_p$, such a point can be obtained by truncating to $1 - \delta$ the magnitude of the reflection coefficients or that of the poles. In problems when no such insight is available or when one does not want to embark in such an analysis, the origin can be used as a starting point.

D. Implementation

The purpose of the section is to provide an implementable version of the conceptual projection method described above. Our algorithm is based on the gradient method with adaptive step size selection proposed in [14]. The unstable vector is denoted by $\boldsymbol{b}$, its best stable approximation by $\boldsymbol{a}$, and the Euclidean norm by $\| \cdot \|$.

1) Fix $\boldsymbol{k}_0$ in $\mathcal{K}_p$, β, δ, and ϵ. Set $n = 0$.
2) Compute $\nabla \Phi(\boldsymbol{k}_n) = \boldsymbol{J}(\boldsymbol{k}_n)\boldsymbol{Q}(\mathcal{L}(\boldsymbol{k}_n) - \boldsymbol{b})$. Set $\lambda_n = 1$.
3) Compute $\Delta = \Phi(\boldsymbol{k}_n - \lambda_n \nabla \Phi(\boldsymbol{k}_n)) - \Phi(\boldsymbol{k}_n) + 0.5\lambda_n \|\nabla \Phi(\boldsymbol{k}_n)\|^2$.
4) If $\Delta > 0$, set $\lambda_n = \beta \lambda_n$ and return to 3).
5) Compute $\boldsymbol{k}_{n+1} = \mathcal{P}(\boldsymbol{k}_n - \lambda_n \nabla \Phi(\boldsymbol{k}_n))$.
6) If $|\Phi(\boldsymbol{k}_n) - \Phi(\boldsymbol{k}_{n+1})| > \epsilon$, set $n = n + 1$ and return to 2).
7) $\boldsymbol{a} = \mathcal{L}(\boldsymbol{k}_{n+1})$. Stop.

The step sizes, $(\lambda_n)_{n \geq 0}$, are determined in loop 3)–4). According to the analysis made in [14], β should be chosen in the interval $]0.5, 0.8[$. Let us also mention that the somewhat tedious derivation of the Jacobian matrix $\boldsymbol{J}$ needed at step 2) can be performed easily with the mathematical symbol manipulator *Mathematica* [17] for arbitrary p.

IV. Simulation Results

Simulations for order p up to 8 have been carried out to test the proposed algorithm. In this section, we present results which are typical of its performance relative to other stabilization methods. Two unstable vectors $\boldsymbol{b}$ are considered, one of dimension $p = 4$ and one of dimension $p = 8$. The following methods are used to stabilize them.

a) Projection onto the (closure of the) sufficiency diamond $\mathcal{S}_p^-$.
b) Pole reflection.
c) Scaling.
d) Truncation of the magnitude of the poles to $1 - \delta$.
e) Truncation of the magnitude of the reflection coefficients to $1 - \delta$.
f) The best approximation algorithm with $\boldsymbol{Q} = \boldsymbol{I}$.

The stabilized filter in a) is obtained by $\boldsymbol{a} = (1/\Sigma_{i=1}^p |b_i|)\boldsymbol{b}$. Method b) is implemented by replacing every pole Z outside the unit circle by its conjugate reciprocal $1/\bar{Z}$ and causes a simple scaling of the magnitude of the frequency response of the system. Method c) consists in finding the largest η in $]0, 1]$ such that $\boldsymbol{a} = \eta \boldsymbol{b}$ is in $\mathcal{S}_p$. From the standpoint of control theory, if the predictor $\boldsymbol{a}$ is in the feedback path, η can be regarded as the feedback gain needed to achieve stability. In d), stability is obtained by replacing every pole Z outside the unit circle by $(1 - \delta)Z/|Z|$. The stabilized filter in e) is given by $\boldsymbol{a} = \mathcal{L}(\mathcal{P}(\mathcal{L}^{-1}(\boldsymbol{b})))$, under the provision that $\mathcal{L}^{-1}(\boldsymbol{b})$ exists. This is a convenient stabilization scheme for lattice filters. Finally f) is the proposed best approximation algorithm. In d)–f), δ was set to 10^{-4}. Moreover, $\beta = 0.75$, $\epsilon = 10^{-5}$, and $\boldsymbol{k}_0 = \boldsymbol{0}$ were used in f). The stable vectors produced by these various methods are displayed in Table II for $p = 4$ and Table III for $p = 8$, where d denotes the Euclidean metric. The best approximation algorithm converged in 15 iterations for $p = 4$ and 16 iterations for $p = 8$.

These simulations confirm that, although computationally simpler, methods a)–e) introduce large perturbations in the coefficients. As a result, one may expect the properties of the original system to have been severely degraded, if not lost, through such stabilization procedures. It should also be clear that methods a)–e) may not be used in lieu of f) in the set theoretic identification of ARMA systems by the successive projections algorithm of [6].

V. Conclusions

A method has been proposed to determine best stable and invertible approximations for ARMA systems, the criterion of proximity being a general quadratic distance in the coefficient space. The al-

TABLE II
STABILIZATION RESULTS – $p = 4$

Method	a_1	a_2	a_3	a_4	$d(\boldsymbol{a}, \boldsymbol{b})$
Unstable	2.100	−3.300	1.700	−0.800	—
(a)	0.266	−0.418	0.215	−0.101	3.79
(b)	1.278	−1.256	0.539	−0.183	2.57
(c)	1.035	−1.627	0.838	−0.394	2.20
(d)	1.614	−1.954	0.942	−0.383	1.67
(e)	1.539	−1.944	1.251	−0.800	1.53
(f)	2.192	−2.992	1.954	−0.794	0.41

TABLE III
STABILIZATION RESULTS – $p = 8$

Method	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$d(\boldsymbol{a}, \boldsymbol{b})$
Unstable	−6.000	−12.00	−10.00	2.000	13.00	10.00	4.000	0.800	—
(a)	−0.104	−0.208	−0.173	0.035	0.225	0.173	0.069	0.014	23.4
(b)	−1.710	−0.949	0.589	1.376	1.037	0.441	0.109	0.012	22.4
(c)	−0.199	−0.398	−0.332	0.066	0.431	0.332	0.133	0.027	23.1
(d)	−2.983	−3.363	−0.831	2.417	3.206	1.847	0.607	0.097	18.5
(e)	0.407	1.921	−1.500	−0.043	1.779	−1.679	−0.686	0.800	24.4
(f)	−5.594	−11.97	−10.35	2.027	11.97	10.35	3.972	0.595	1.23

gorithm proceeds by steepest descent in the stability hypercube of the reflection space and was seen to converge efficiently to a solution under mild conditions. Besides being needed for ARMA set theoretic identification by successive projections, the method is also of interest in problems where it is needed to stabilize and render invertible a system while perturbing the least its coefficients. Thus, it finds applications in areas such as time series analysis, control, system identification, spectral estimation, and linear prediction.

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Symmetric Solutions and Eigenvalue Problems of Toeplitz Systems

Dawei Huang

***Abstract*—Algorithms and properties of symmetric solutions of a Toeplitz system are studied and applied to an eigenvalue problem and Pisarenko's method.**

I. INTRODUCTION

The symmetric Toeplitz matrix has been studied and applied in digital signal processing extensively. A very important reason is the popular Levinson recursion. Some recent works relevant to the Levinson algorithm involve the minimum eigenvalue–eigenvector of the Toeplitz matrix [1], [2], which suggest fast algorithms for

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