Convex Multiresolution Analysis

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Abstract—A standard wavelet multiresolution analysis can be defined via a sequence of projectors onto a monotone sequence of closed vector subspaces possessing certain properties. We propose a nonlinear extension of this framework in which the vector subspaces are replaced by convex subsets. These sets are chosen so as to provide a recursive, monotone approximation scheme that allows for various signal and image features to be investigated. Several classes of convex multiresolution analyses are discussed and numerical applications to signal and image-processing problems are demonstrated.

Index Terms—Multiresolution analysis, convex sets, hierarchical signal analysis, nonlinear filter banks, projections.

1 Introduction

The concept of multiresolution analysis has found numerous applications in a wide array of signal and image processing problems. The tenet of this hierarchical framework is to study a signal x in $\mathrm{L}^2(\mathbb{R}^n)$ at monotonically coarser resolutions $(x_j)_{j\in\mathbb{Z}}$. In early work, $(x_j)_{j\in\mathbb{Z}}$ was constructed by convolving x with increasingly broad Gaussian functions, e.g., [1], [7]. Another approach is to obtain the lower resolution approximations $(x_j)_{j\in\mathbb{Z}}$ to x as its projections onto a suitable monotone sequence of closed vector subspaces [18], [20]. This effort led to the construction of wavelet bases in $\mathrm{L}^2(\mathbb{R}^n)$.

In the above methods, the perceptual notion of resolution is closely related to that of local smoothness. The low-resolution approximations are obtained by linear processing, which typically produces a uniform blurring of the signal through which important features cannot be preserved, e.g., singularities in the case of 1D signals or edges in the case of images. This shortcoming prompted the development of alternative multiresolution analysis techniques based for instance on morphology [3], nonlinear digital filtering [2], [16], [23], and partial differential equations [21].

In this paper, we explore an alternative approach to nonlinear multiresolution signal analysis. More specifically, we present an operator theoretic formulation which extends the standard linear projection framework to more general, nonlinear, recursive signal approximation schemes in a Hilbert space $\mathcal{H}$. In the proposed Convex Multiresolution Analysis (CMRA) framework, a lower approximation x_j to a signal x is obtained by projecting the higher approximation x_{j-1} onto a closed and convex subset C_j of $\mathcal{H}$. In this context, the word "resolution" is employed in a broad sense, as the information lost in passing from "scale" $j-1$ to "scale" j is not necessarily associated with a uniform reduction of the finer details. In simple terms, a CMRA acts as a nonlinear magnifying device that allows us to investigate a wide range of signal features at various scales. The generality and flexibility of CMRAs stem from their simple geometric construction from sequences of nested convex sets. It should be noted that convex sets have already been used—for purposes different from ours—as an efficient tool to represent signal properties in various signal analysis and processing problems, e.g., [8], [12], [13], [19], [22].

The remainder of the paper consists of four sections. The theoretical foundations of convex multiresolution analysis are laid in Section 2. General examples of CMRAs are then developed in Section 3 and simulation results are presented in Section 4. Concluding remarks appear in Section 5.

2 Convex Multiresolution Analysis (CMRA)

The signal space is a real Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot|\cdot\rangle$, norm $\|\cdot\|$, and distance d. The projection of a signal $x\in\mathcal{H}$ onto a nonempty closed and convex set $C_j\subset\mathcal{H}$ is the unique signal $P_jx\in C_j$ that satisfies $\|x-P_jx\| = \inf_{y\in C_j}\|x-y\| \triangleq d(x,C_j)$. The projector P_j is characterized by the variational inequality [6]

$$(\forall x\in\mathcal{H})(\forall y\in C_j)\quad \langle y-P_jx \mid x-P_jx\rangle \le 0. \tag{1}$$

In the Euclidean plane, the geometrical interpretation of this inequality is that the angle between the vectors $y-P_jx$ and $x-P_jx$ is obtuse (cf. Fig. 1).

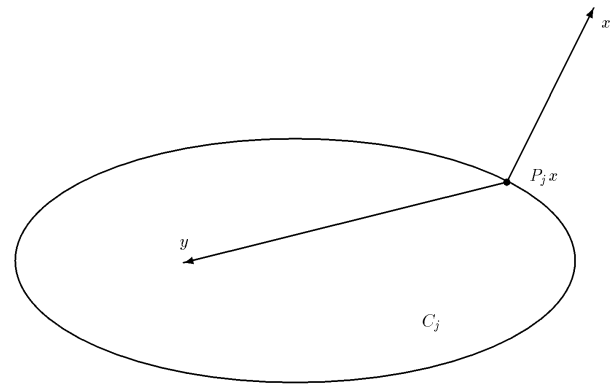


Fig. 1. Geometrical interpretation of property (1).

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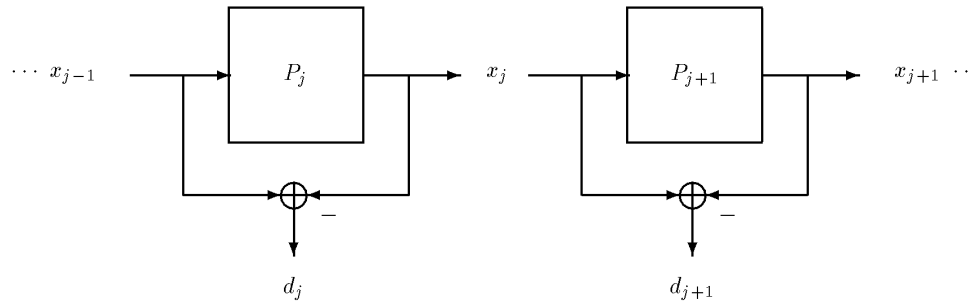


Fig. 2. Implementation of a CMRA processor.

Our framework is built around the two definitions below.

DEFINITION 1. *A multiresolution approximation to a signal $x \in \mathcal{H}$ is a sequence $(x_j)_{j\in\mathbb{Z}} \subset \mathcal{H}$ such that*

(i) $(\forall j \in \mathbb{Z})\ \|x - x_{j-1}\| \leq \|x - x_j\|$.
(ii) $\lim_{j\to-\infty} x_j = x$.
(iii) $(\exists x_{+\infty} \in \mathcal{H})\ \lim_{j\to+\infty} x_j = x_{+\infty}$.

The signal x_j is called an approximation to x at resolution level j. The sequence $(d_j)_{j\in\mathbb{Z}}$ given by

$$(\forall j \in \mathbb{Z})\ d_j = x_{j-1} - x_j \tag{2}$$

is called the detail sequence of the multiresolution approximation.

Loosely speaking, $(x_j)_{j\in\mathbb{Z}}$ is a monotone sequence of successive approximations to x. The signal $x_{+\infty}$ is the limiting coarse approximation to x and the signal d_j represents the details lost in passing from resolution level $j-1$ to resolution level j.

DEFINITION 2. *A convex multiresolution analysis (CMRA) of $\mathcal{H}$ is a sequence $(P_j)_{j\in\mathbb{Z}}$ of projectors onto closed and convex subsets $(C_j)_{j\in\mathbb{Z}}$ of $\mathcal{H}$ that satisfy the following properties:*

(a) $(\forall j \in \mathbb{Z})\ \ C_j \subset C_{j-1}$.
(b) $\overline{\lim_{j\to-\infty} C_j} = \mathcal{H}$.
(c) $C \triangleq \lim_{j\to+\infty} C_j \neq \varnothing$.
(d) $(\forall j \in \mathbb{Z})\ P_j = P_j \circ P_{j-1}$.

The sequence $(P_j)_{j\in\mathbb{Z}}$ is called a CMRA of $\mathcal{H}$ relative to $(C_j)_{j\in\mathbb{Z}}$.

In words, $(C_j)_{j\in\mathbb{Z}}$ is a decreasing (cf. (a)) sequence of convex sets which is complete (cf. (b)) and nondegenerate (cf. (c)). It can be shown that, under assumption (a), the nondegeneracy condition (c) holds if and only if there exists a signal $z \in \mathcal{H}$ such that $\sup_{j\in\mathbb{Z}} d(z, C_j) < +\infty$ [6].[1] In view of (d), P_j can be regarded as a scale transition operator and the successive approximations to a signal x are constructed recursively: $P_j x = P_j \circ P_{j-1} x$. It is noteworthy that the conditions of "fidelity" and "causality" found in the definition of the multiscale analysis of [21] are akin to conditions (b) and (d).

We now verify that Definition 2 is indeed compatible with Definition 1.

PROPOSITION 1. *Let $(P_j)_{j\in\mathbb{Z}}$ be a CMRA of $\mathcal{H}$, $x \in \mathcal{H}$, and $(\forall j \in \mathbb{Z})\ x_j = P_j x$. Then $(x_j)_{j\in\mathbb{Z}}$ is a multiresolution approximation to x and $x_{+\infty}$ is the projection of x onto C.*

PROOF. (i) in Definition 1 follows from (a). It also follows from (a) that $(P_j x)_{j\in\mathbb{Z}}$ converges to the projection of x onto $\overline{\bigcup_{j\in\mathbb{Z}} C_j}$ as $j \to -\infty$ [6] which, by virtue of (b), establishes (ii) in Definition 1. Finally, (a) and (c) imply that $(P_j x)_{j\in\mathbb{Z}}$ converges to the projection of x onto $\bigcap_{j\in\mathbb{Z}} C_j$ as $j \to +\infty$ [6]. This proves (iii) in Definition 1 and the last assertion. □

The practical significance of the above proposition is that CMRAs are simple and powerful devices to generate multiresolution approximations. It follows from the recursivity condition (d) that a CMRA can be implemented via a cascade of (possibly) nonlinear filters, as shown in Fig. 2. At each resolution level j, two signals are generated, namely the approximation x_j and the detail signal d_j. These signals are related via the following reconstruction formula.

PROPOSITION 2. $(\forall j \in \mathbb{Z})\ x = x_j + \sum_{i\leq j} d_i$.

PROOF. Fix $j \in \mathbb{Z}$. For every integer $k < j$, we have $\sum_{i=k+1}^{j} d_i = \sum_{i=k+1}^{j} x_{i-1} - x_i = x_k - x_j$. Taking the limit as $k \to -\infty$, (ii) yields $\sum_{i\leq j} d_i = x - x_j$, which proves the announced result. □

It should also be noted that, if $x \in C_j$, then $x_i = x$ for every $i \leq j$. A geometrical illustration of a CMRA is provided in Fig. 3.

3 CONSTRUCTION OF CMRAS

In this section, we construct several classes of CMRAs. Throughout, $(C_j)_{j\in\mathbb{Z}}$ is a sequence of closed and convex subsets of $\mathcal{H}$ with projectors $(P_j)_{j\in\mathbb{Z}}$. In addition, x is an arbitrary signal in $\mathcal{H}$, $(x_j)_{j\in\mathbb{Z}}$ the associated multiresolution approximation, i.e., $(\forall j \in \mathbb{Z})\ x_j = P_j x$, and $(d_j)_{j\in\mathbb{Z}}$ its detail sequence, as defined in (2). $\mathbb{N}^*$ is the set of positive integers,

1. In connection with Definition 2, it is important to stress that a sequence of closed and convex sets $(C_j)_{j\in\mathbb{Z}}$ that satisfies (a), (b), and (d) needs not satisfy (c). A simple counterexample is obtained by taking $\mathcal{H} = \mathbb{R}$ and $(\forall j \in \mathbb{Z})\ C_j = [j, +\infty[$.

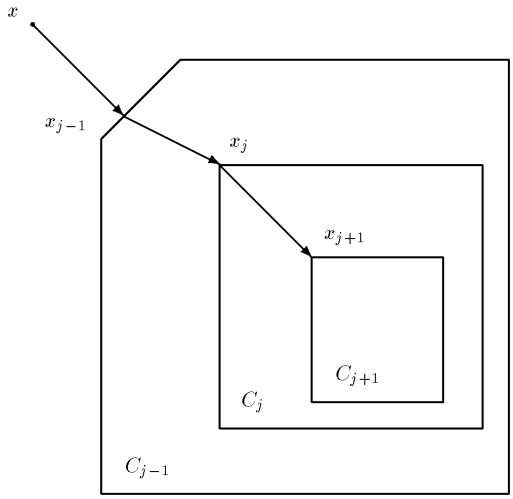


Fig. 3. Geometrical illustration of a CMRA.

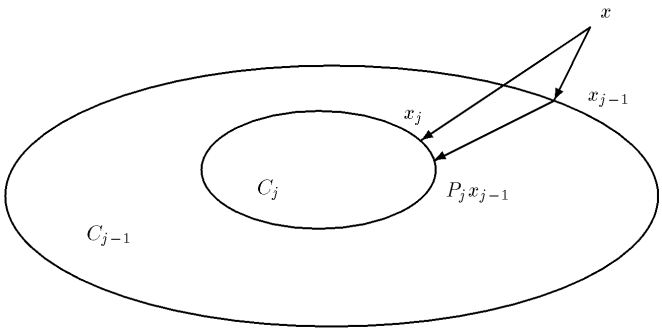


Fig. 4. Violation of condition (d).

$\mathbb{R}_+$ the set of nonnegative reals, $\mathbb{R}_+^*$ the set of positive reals, and $y \overset{+}{\propto} z$ means that $y = \alpha z$ for some $\alpha \in \mathbb{R}_+$. bd C_j is the boundary of C_j and

$$N_j(y) = \{t \in \mathcal{H} \mid (\forall z \in C_j)\ \langle z - y \mid t\rangle \le 0\} \tag{3}$$

is the normal cone to C_j at $y \in C_j$ ($N_j(y) = \{0\}$ if y lies in the interior of C_j).

In connection with Definition 2, it is important to observe that, if $(C_j)_{j\in\mathbb{Z}}$ is a sequence of vector subspaces, then (d) is equivalent to (a) in Definition 2 [6]. Unfortunately, (d) no longer follows from (a) in the nonlinear case, as shown in Fig. 4. Thus, a difficulty one faces in building a CMRA is to find a sequence $(C_j)_{j\in\mathbb{Z}}$ of convex sets that comply with (d). Our constructions will hinge on the following fact (cf. Fig. 5).

PROPOSITION 3. *Suppose that $(C_j)_{j\in\mathbb{Z}}$ satisfies (a)-(c). Then $(P_j)_{j\in\mathbb{Z}}$ is a CMRA of $\mathcal{H}$ if*

$$(\forall j \in \mathbb{Z})\quad d_j \in N_j(x_j). \tag{4}$$

PROOF. According to Definition 2, we must verify that (d) holds. To this end, fix $j \in \mathbb{Z}$. Then we must show $P_j x = P_j \circ P_{j-1} x$, i.e., with our notations, $x_j = P_j x_{j-1}$. In view of (2) and (3), (4) can be rewritten as

$$x_{j-1} - x_j \in \{t \in \mathcal{H} \mid (\forall y \in C_j)\ \langle y - x_j \mid t\rangle \le 0\}. \tag{5}$$

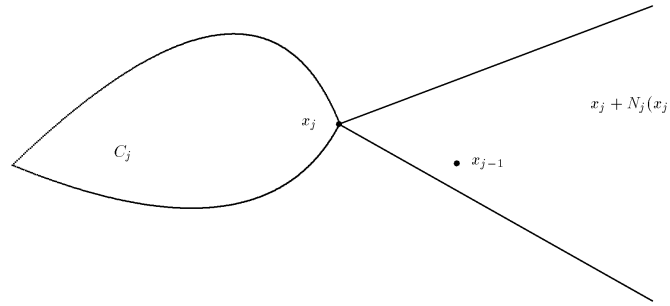


Fig. 5. Illustration of condition (4).

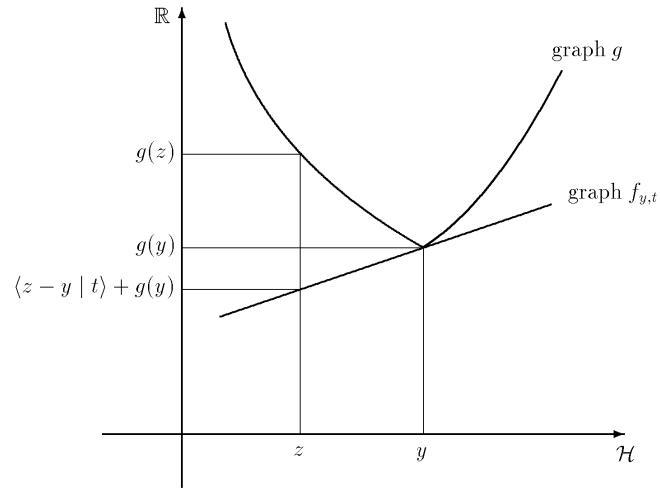


Fig. 6. Graphical interpretation of subgradients.

Consequently, we obtain

$$(\forall y \in C_j)\ \langle y - x_j \mid x_{j-1} - x_j\rangle \le 0. \tag{6}$$

However, since $x_j \in C_j$, this variational inequality characterizes x_j as the projection of x_{j-1} onto C_j by virtue of (1). Thus, $x_j = P_j x_{j-1}$, as desired. □

3.1 CMRA Induced by a Convex Function

Let $g : \mathcal{H} \to \mathbb{R}$ be a convex and continuous functional. The subdifferential of g at $y \in \mathcal{H}$ is the (nonempty) set of its subgradients, i.e., [15]

$$\partial g(y) = \{t \in \mathcal{H} \mid (\forall z \in \mathcal{H})\ \langle z - y \mid t\rangle + g(y) \le g(z)\}. \tag{7}$$

Graphically, t is a subgradient of g at y if the continuous affine functional $f_{y,t} : z \mapsto \langle z - y \mid t\rangle + g(y)$, which has "slope" t and takes the same value as g at y, minorizes g on $\mathcal{H}$ (cf. Fig. 6). In particular, if g is (Gâteaux) differentiable at y, then the gradient $\nabla g(y)$ is the unique subgradient of g at y [15].

PROPOSITION 4. *Let $g : \mathcal{H} \to \mathbb{R}$ be a convex and continuous functional, let $z \in \mathcal{H}$, and let $(\delta_j)_{j\in\mathbb{Z}}$ be a decreasing sequence in $]g(z), +\infty[$ that diverges as $j \to -\infty$. Define*

$$(\forall j \in \mathbb{Z})\ C_j = \{y \in \mathcal{H} \mid g(y) \le \delta_j\} \tag{8}$$

and suppose

$$(\forall j \in \mathbb{Z})\left(\exists t \in \partial g\left(x_j\right)\right)\ t \overset{+}{\propto} d_j. \tag{9}$$

Then $(P_j)_{j\in\mathbb{Z}}$ is a CMRA of $\mathcal{H}$.

PROOF. The convexity of C_j in (8) follows from that of g and its closedness from the continuity of g. In addition, it is an easy matter to see that the assumptions on $(\delta_j)_{j\in\mathbb{Z}}$ secure that conditions (a) and (b) hold. Now let $S = \{y \in \mathcal{H} \mid g(y) \le g(z)\}$. Then (c) is satisfied as $\lim_{j\to+\infty} C_j \supset S \neq \emptyset$. In view of Proposition 3, it remains to show that (4) holds. If $x \in C_j$, then $x = x_j = x_{j-1}$ and consequently $d_j = 0 \in N_j(x_j)$. Next, we assume $x \notin C_j$, which implies $x_j \in \operatorname{bd} C_j$. In view of (9), we can find $t \in \partial g(x_j)$ such that $t \overset{+}{\propto} d_j$ and, consequently, it suffices to prove $t \in N_j(x_j)$. To this end, fix an arbitrary $y \in C_j$. Then $g(y) \le \delta_j = g(x_j)$ and it follows from (7) that $\langle y - x_j \mid t\rangle \le g(y) - g(x_j) \le 0$. This shows $t \in N_j(x_j)$, as desired. □

As discussed in [12], a broad spectrum of signal properties can be translated into inequality constraints of the form "$g(y) \le 0$," where g is as above. In this light, the set C_j defined in (8) represents the class of signals that satisfy a weakened or strengthened version of this constraint, according as $\delta_j \ge 0$ or $\delta_j \le 0$.

3.2 CMRA Induced by a Convex Set

PROPOSITION 5. *Let P_S denote the projector onto a nonempty closed and convex subset S of $\mathcal{H}$, let $(\delta_j)_{j\in\mathbb{Z}}$ be a decreasing sequence in $\mathbb{R}_+^*$ that diverges as $j \to -\infty$, and let*

$$(\forall j \in \mathbb{Z})\ C_j = \{y \in \mathcal{H} \mid d(y, S) \le \delta_j\}. \tag{10}$$

Then $(P_j)_{j\in\mathbb{Z}}$ is a CMRA of $\mathcal{H}$.

PROOF. Let $g : y \mapsto d(y, S)$. Then g is continuous and, by the convexity of S, convex. Consequently, (a)-(c) hold as in the proof of Proposition 4. According to Proposition 3, it remains to show that (4) is fulfilled. Let p be the projection of the analyzed signal x onto S. Then it is sufficient to show that the multiresolution approximation $(x_j)_{j\in\mathbb{Z}}$ is aligned on the segment between x and p, i.e., $(\forall j \in \mathbb{Z})\ x_j \in [p, x]$. We have [12]

$$\big(\forall y \in \mathcal{H} \setminus S\big)\ \nabla g(y) = \frac{y - P_S y}{d(y, S)} \overset{+}{\propto} y - P_S y. \tag{11}$$

Now let j be any integer for which $x \notin C_j$ and let y_j be the unique point in $[p, x] \cap \operatorname{bd} C_j$ (if $x \in C_j$, we trivially have $x_j = x \in [p, x]$). We shall show $x_j = y_j$, which will prove the assertion. Since $y_j \in C_j$, (1) yields

$$\langle x_j - y_j \mid x_j - x\rangle \le 0. \tag{12}$$

On the other hand, (7) implies $\langle x_j - y_j \mid \nabla g(y_j)\rangle \le g(x_j) - g(y_j) = 0$. It then follows from (11) that $\langle x_j - y_j \mid y_j - P_S y_j\rangle \le 0$. However, $y_j \in [p, x]$. Hence, $P_S y_j = p$ and therefore $\langle x_j - y_j \mid y_j - p\rangle \le 0$. Moreover, $y_j - p \overset{+}{\propto} x - y_j$, and, hence,

$$\langle x_j - y_j \mid x - y_j\rangle \le 0. \tag{13}$$

Adding (12) and (13) yields $\|x_j - y_j\|^2 \le 0$, i.e., $x_j = y_j$. □

As a by-product of the above proof, it emerges that the multiresolution approximation $(x_j)_{j\in\mathbb{Z}}$ lies on the segment between the analyzed signal, x, and its best approximation, p, from S. More precisely,

$$(\forall j \in \mathbb{Z})\ \ x_j = \alpha_j x + (1 - \alpha_j)p$$

where

$$\alpha_j = \begin{cases} \delta_j / d(x, S) & \text{if } \delta_j < d(x, S) \\ 1 & \text{otherwise} \end{cases} \tag{14}$$

3.3 Polyhedral Multiresolution Analysis (PMRA)

Let $U = \{y \in \mathcal{H} \mid \|y\| = 1\}$ be the unit sphere in $\mathcal{H}$. It is assumed that $(C_j)_{j\in\mathbb{Z}}$ is a sequence of polyhedra, namely

$$(\forall j \in \mathbb{Z})\, C_j = \bigcap_{i=1}^{m_j} \Big\{ y \in \mathcal{H} \,\Big|\, \big\langle y \,\big|\, b_{i,j} \big\rangle \le \delta_{i,j} \Big\}, \tag{15}$$

where $m_j \in \mathbb{N}^*$, $\big(b_{i,j}\big)_{1\le i\le m_j} \subset U$, and $\big(\delta_{i,j}\big)_{1\le i\le m_j} \subset \mathbb{R}$. In words, each set C_j is a finite intersection of closed affine half-spaces and, as such, closed and convex. A CMRA relative to (15) will be called a polyhedral multiresolution analysis (PMRA).

PROPOSITION 6. *Let $(C_j)_{j\in\mathbb{Z}}$ be a sequence as in (15) satisfying (a)-(c). Then $(P_j)_{j\in\mathbb{Z}}$ is a PMRA of $\mathcal{H}$ if*

$$(\forall j \in \mathbb{Z})\ x \notin C_j \ \Rightarrow$$

$$\Big(\exists \big(\alpha_{i,j}\big)_{i\in I_j} \subset \mathbb{R}_+\Big) d_j = \sum_{i\in I_j} \alpha_{i,j} b_{i,j}, \tag{16}$$

where $I_j = \{1 \le i \le m_j \mid \langle x_j \mid b_{i,j}\rangle = \delta_{i,j}\}$.

PROOF. Fix $j \in \mathbb{Z}$. According to Proposition 3, we must show $d_j \in N_j(x_j)$. If $x \in C_j$, then $x_j = x_{j-1}$ and $d_j = 0 \in N_j(x_j)$. Now assume $x \notin C_j$, which implies $x_j \in \operatorname{bd} C_j$. In view of the definition of I_j, the active hyperplanes at x_j are

$$\Big(\Big\{ y \in \mathcal{H} \,\Big|\, \big\langle y \,\big|\, b_{i,j} \big\rangle = \delta_{i,j} \Big\} \Big)_{i\in I_j}$$

and the tangent cone to C_j at x_j is

$$T_j\big(x_j\big) = \cap_{i\in I_j} \Big\{ y \in \mathcal{H} \,\Big|\, \big\langle y \,\big|\, b_{i,j} \big\rangle \le 0 \Big\}.$$

However, $N_j(x_j)$ is the polar cone of $T_j(x_j)$ which, by Farkas' lemma [9], consists of all the linear combinations of the vectors $\big(b_{i,j}\big)_{i\in I_j}$ with nonnegative coefficients. Hence, (16) implies $d_j \in N_j(x_j)$ and the proof is complete.

An important special case is described next. □

PROPOSITION 7. *Given $m \in \mathbb{N}^*$, $(\rho_i)_{1\le i\le m} \subset \mathbb{R}$, and $(b_i)_{1\le i\le m} \subset U$, let $\langle y \mid b_i\rangle \le \rho_i$ $(1 \le i \le m)$ be a consistent system of m nonredundant linear inequalities. Define*

$$(\forall j \in \mathbb{Z})\, C_j = \bigcap_{i=1}^{m} \left\{ y \in \mathcal{H} \,\middle|\, \left\langle y \middle| b_i \right\rangle \le \rho_i + \delta_j \right\}, \quad (17)$$

where the decreasing sequence $\left(\delta_j\right)_{j\in\mathbb{Z}} \subset \mathbb{R}_+^*$ *diverges as* $j \to -\infty$. *Then* $(P_j)_{j\in\mathbb{Z}}$ *is a PMRA of* $\mathcal{H}$.

PROOF. Geometrically, $(C_j)_{j\in\mathbb{Z}}$ is constructed as follows. The nonempty polyhedron $S = \{y \in \mathcal{H} \mid \langle y \mid b_i\rangle \le \rho_i\}$ is delimited by m hyperplanes $H_i = \{y \in \mathcal{H} \mid \langle y \mid b_i\rangle = \rho_i\}$ $(1 \le i \le m)$. The polyhedron C_j in (17) is obtained by performing a translation of length δ_j on each H_i in the direction of its outward normal b_i. In light of the properties of the sequence $(\delta_j)_{j\in\mathbb{Z}}$, one easily sees that conditions (a)-(c) are fulfilled. Now fix $j \in \mathbb{Z}$ and assume $x \notin C_j$. According to Proposition 6, it remains to show

$$\left(\exists \left(\alpha_{i,j}\right)_{i\in I_j} \subset \mathbb{R}_+\right) d_j = \sum_{i\in I_j} \alpha_{i,j} b_i$$

where

$$I_j = \{1 \le i \le m \mid \langle x_j \mid b_i\rangle = \rho_i + \delta_j\}. \quad (18)$$

Since $x_j = P_j x$, we have $x - x_j \in N_j(x_j)$, i.e., $x - x_j = \sum_{i\in I_j} \beta_{i,j} b_i$ for some $\left(\beta_{i,j}\right)_{i\in I_j} \subset \mathbb{R}_+$. Likewise, at level $j-1$, we can write $x - x_{j-1} = \sum_{i\in I_{j-1}} \beta_{i,j-1} b_i$ for some $\left(\beta_{i,j-1}\right)_{i\in I_{j-1}} \subset \mathbb{R}_+$. However, by construction, $I_{j-1} \subset I_j$ and $(\forall i \in I_{j-1})\ \beta_{i,j-1} \le \beta_{i,j}$. Hence,

$$d_j = \sum_{i\in I_{j-1}} \left(\beta_{i,j} - \beta_{i,j-1}\right) b_i + \sum_{i\in I_j \setminus I_{j-1}} \beta_{i,j} b_i, \quad (19)$$

and we obtain (18) with $\alpha_{i,j} = \beta_{i,j}$ if $i \in I_j \setminus I_{j-1}$, and $\alpha_{i,j} = \beta_{i,j} - \beta_{i,j-1}$ else. □

3.4 Linear Multiresolution Analysis (LMRA)

Let $(C_j)_{j\in\mathbb{Z}}$ be a decreasing sequence of closed vector subspaces whose union is dense in $\mathcal{H}$. Then conditions (a)-(d) are satisfied [6] and the CMRA thus obtained will be called a linear multiresolution analysis (LMRA). Moreover, (4) becomes $(\forall j \in \mathbb{Z})\ \langle x_j \mid d_j\rangle = 0$. In words, the detail signal is orthogonal to the approximation at each resolution level.

A well-known special instance of LMRA is found when $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$ and when the two additional assumptions below are made.

(e) $C = \{0\}$.

(f) There exists a so-called scaling function φ such that, for every integer j, $(D^j T^k \varphi)_{k\in\mathbb{Z}}$ is an orthonormal basis of C_j, where $T^k : y \mapsto y(\cdot - k)$ is a translation operator and $D^j : y \mapsto 2^{-j/2} y(2^{-j}\cdot)$ a unitary dyadic dilation operator.

We then obtain the popular wavelet framework of [17], [20]. In this context, the term "multiresolution analysis" is used because the subspaces $(C_j)_{j\in\mathbb{Z}}$ are obtained by scaling the reference space C_0 in the sense that

$$(\forall (y, j) \in \mathcal{H} \times \mathbb{Z})\ \ y \in C_0 \Leftrightarrow D^j y \in C_j. \quad (20)$$

Hence, the distinguishability of small close objects or details in x_j decreases as $j \to +\infty$ (increasingly locally smooth approximations are obtained). An important property of this framework is that the multiresolution approximation $(x_j)_{j\in\mathbb{Z}}$ can be efficiently computed using decimated filter banks, e.g., [14].

Let us note that our setting also includes alternative LMRAs that do not fulfill the scaling condition (20), e.g., the nonstationary multiscale analysis of [10], as well as multiresolution analyses based on frames [4].

3.5 Product CMRA

Let I be a finite index set, let $(\mathcal{H}_i)_{i\in I}$ be a family of real Hilbert spaces with respective norms $(\|\cdot\|_i)_{i\in I}$, and let the signal space be defined as the cartesian product $\mathcal{H} = \bigtimes_{i\in I} \mathcal{H}_i$. The generic signal x can then be written as $x = (x^i)_{i\in I}$, where $(\forall i \in I)\ x^i \in \mathcal{H}_i$. The canonical hilbertian norm of a signal $x \in \mathcal{H}$ is $\|x\| = \left(\sum_{i\in I} \left\|x^i\right\|_i^2\right)^{1/2}$.

PROPOSITION 8. *For every* $i \in I$, *let* $\left(P_j^i\right)_{j\in\mathbb{Z}}$ *be a CMRA of* $\mathcal{H}_i$ *relative to* $\left(C_j^i\right)_{j\in\mathbb{Z}}$ *and define*

$$(\forall j \in \mathbb{Z})\ C_j = \bigtimes_{i\in I} C_j^i. \quad (21)$$

Then

(j) $(\forall j \in \mathbb{Z})\ P_j x = \left(P_j^i x_i\right)_{i\in I}$;

(jj) $(P_j)_{j\in\mathbb{Z}}$ is a CMRA of $\mathcal{H}$ relative to $(C_j)_{j\in\mathbb{Z}}$.

PROOF. To prove (j), we first observe that, by virtue of (21), $(C_j)_{j\in\mathbb{Z}}$ is indeed a sequence of closed and convex subsets of $\mathcal{H}$. Now fix $j \in \mathbb{Z}$. Then

$$\begin{aligned} \left\|x - P_j x\right\|^2 &= \min_{y\in C_j} \|x - y\|^2 \\ &= \min_{\left(y^i\right)_{i\in I} \in \bigtimes_{i\in I} C_j^i} \sum_{i\in I} \left\|x^i - y^i\right\|_i^2 \\ &= \sum_{i\in I} \left\|x^i - P_j^i x^i\right\|_i^2 \\ &= \left\|x - \left(P_j^i x^i\right)_{i\in I}\right\|^2. \end{aligned} \quad (22)$$

Next, to prove (jj), we need to check that conditions (a)-(d) in Definition 2 are fulfilled. (a) follows at once from the monotonicity of $\left(C_j^i\right)_{j\in\mathbb{Z}}$ for each $i \in I$. (b) follows from the chain of identities

$$\begin{aligned} \overline{\lim_{j\to-\infty} C_j} &= \overline{\lim_{j\to-\infty} \bigtimes_{i\in I} C_j^i} \\ &= \overline{\bigtimes_{i\in I} \lim_{j\to-\infty} C_j^i} \\ &= \bigtimes_{i\in I} \overline{\lim_{j\to-\infty} C_j^i} \\ &= \bigtimes_{i\in I} \mathcal{H}_i = \mathcal{H}. \end{aligned} \quad (23)$$

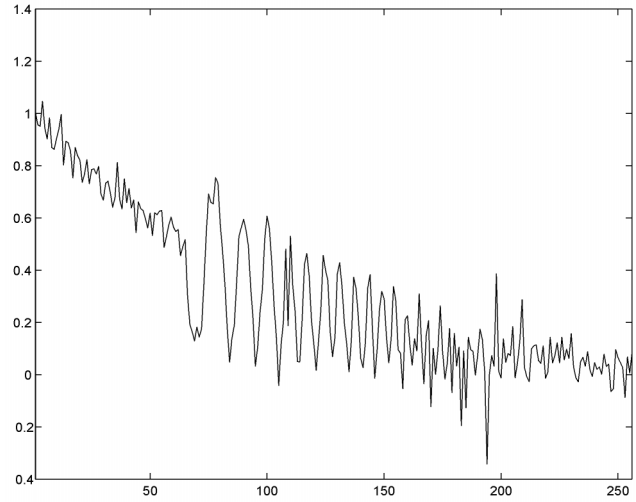


Fig. 7. Analyzed signal.

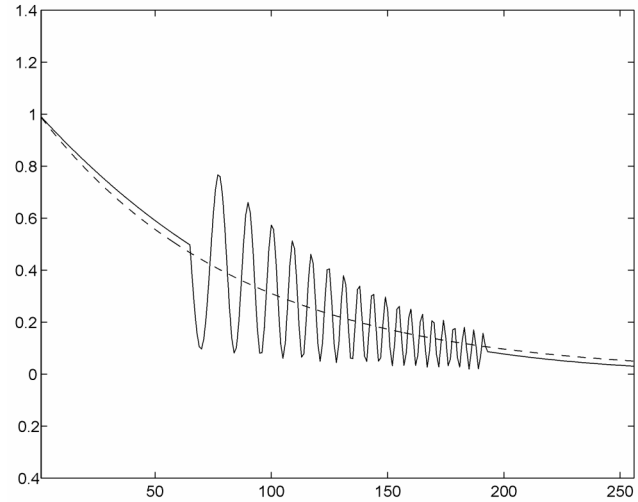


Fig. 8. Original signal (solid line) and center z of the sets (dashed line).

(c): Let $C = \bigtimes_{i\in I} C^i$ where $(\forall i \in I)\ C^i \triangleq \lim_{j\to+\infty} C^i_j \neq \varnothing$. Then $C = \lim_{j\to+\infty} C_j \neq \varnothing$. (d): Fix $j \in \mathbb{Z}$. Since, for each $i \in I$ $\left(P^i_j\right)_{j\in\mathbb{Z}}$ is a CMRA of $\mathcal{H}_i$, (j) yields $P_j x = \left(P^i_j x^i\right)_{i\in I} = \left(P^i_j \circ P^i_{j-1} x^i\right)_{i\in I} = \left(P^i_j\left[P_{j-1}x\right]^i\right)_{i\in I} = P_j \circ P_{j-1}x$ which completes the proof. □

This product space framework is quite versatile as it allows for a decomposition of the signal x into elementary components $(x^i)_{i\in I}$ on which different CMRAs can be performed. At each scale j of a product CMRA, the approximation x_j is constructed as a sum of orthogonal signals in $\mathcal{H}$. To see this let, for every $i \in I$, π_i be the canonical mapping from $\mathcal{H}_i$ into $\mathcal{H}$, i.e.,

$$\left(\forall u \in \mathcal{H}_i\right)\left(\forall k \in I\right)\left[\pi_i(u)\right]^k = \begin{cases} u & \text{if } k = i \\ 0 & \text{else} \end{cases} \tag{24}$$

Then it follows that the signals $\left(\pi_i\left(x^i_j\right)\right)_{i\in I}$ are orthogonal in $\mathcal{H}$ and, moreover, that $x_j = \left(x^i_j\right)_{i\in I} = \sum_{i\in I} \pi_i\left(x^i_j\right)$. Actually, upon defining $(\forall i \in I)\ H_i = \pi_i(\mathcal{H}_i)$, $\mathcal{H}$ can be written as the direct hilbertian sum $\mathcal{H} = \oplus_{i\in I} H_i$ and $\pi_i\left(x^i_j\right)$ is the projection of x_j onto H_i.

Finally, let us note that product CMRAs can also be developed when I is countably infinite. In such instances, the signal space will be defined as

$$\mathcal{H} = \left\{\left(x^i\right)_{i\in I} \in \bigtimes_{i\in I} \mathcal{H}_i \;\middle|\; \sum_{i\in I} \left\|x^i\right\|_i^2 < +\infty\right\}. \tag{25}$$

4 Simulation Results

4.1 Product CMRA of a One-Dimensional Signal

The signal x to be analyzed is the N-point ($N = 256$) signal shown in Fig. 7. It is a noisy measurement of the original signal h shown in Fig. 8, which is the sum of an exponential-like signal and of an oscillatory signal s. This type of signal model frequently arises in respiratory sound analysis. Typically, no prior knowledge about the noise is available in such contexts. In the present simulations, the noise samples are independent and identically distributed and the underlying probability density function is a mixture of two Gaussians. Although the results presented below may depend on the nature of the noise, it is considered as unknown since none of its statistical characteristics have been exploited. It is known, however, that s has support $\mathcal{N}$ of length L (here, $\mathcal{N} = \{65, \ldots, 192\}$) in the time domain. In addition, the L-point DFT $\hat{s}_{\mathcal{N}}$ of the restriction of s to $\mathcal{N}$ (i.e., $(\forall k \in \{0, \ldots, L-1\})\ \hat{s}_{\mathcal{N}}(k) = \sum_{n\in\mathcal{N}} s(n)\exp\left(-i2\pi nk/L\right)$) is known to have support $\mathcal{K}$ (here, $\mathcal{K} = \{3, \ldots, 22\} \cup \{106, \ldots, 125\}$). The signal space $\mathcal{H}$ is the usual N-dimensional Euclidean space. To capture the above properties, a product CMRA with three components is constructed, namely $\mathcal{H} = H_1 \oplus H_2 \oplus H_3$. Let us denote by $\complement\mathcal{A}$ the complement of a set $\mathcal{A}$ and by $1_{\mathcal{A}}$ its characteristic function, i.e., $1_{\mathcal{A}}(n) = 1$ if $n \in \mathcal{A}$ and $1_{\mathcal{A}}(n) = 0$ if $n \in \complement\mathcal{A}$. The subspace

$$H_1 = \{y \in \mathcal{H} \mid y1_{\mathcal{N}} = 0\} \tag{26}$$

excludes the oscillatory part of the signal and the subspace

$$H_2 = \left\{y \in \mathcal{H} \;\middle|\; y1_{\complement\mathcal{N}} = 0 \text{ and } \hat{y}_{\mathcal{N}} 1_{\complement\mathcal{K}} = 0\right\} \tag{27}$$

isolates the frequency band $\mathcal{K}$. The subspace

$$H_3 = \left\{y \in \mathcal{H} \;\middle|\; y1_{\complement\mathcal{N}} = 0 \text{ and } \hat{y}_{\mathcal{N}} 1_{\mathcal{K}} = 0\right\} \tag{28}$$

completes the decomposition. This decomposition can be viewed as a rudimentary tiling of the time-frequency plane. In each factor space $\mathcal{H}_i$, a CMRA relative to the balls

$$\left(\forall j \in \mathbb{Z}\right) C^i_j = \left\{y \in \mathcal{H}_i \;\middle|\; \left\|y - z^i\right\|_i \leq \delta_{i,j}\right\} \tag{29}$$

is employed, where the decreasing sequence $\left(\delta_{i,j}\right)_{j\in\mathbb{Z}} \subset \mathbb{R}^*_+$ diverges as $j \to -\infty$. We thus obtain a (product) CMRA by

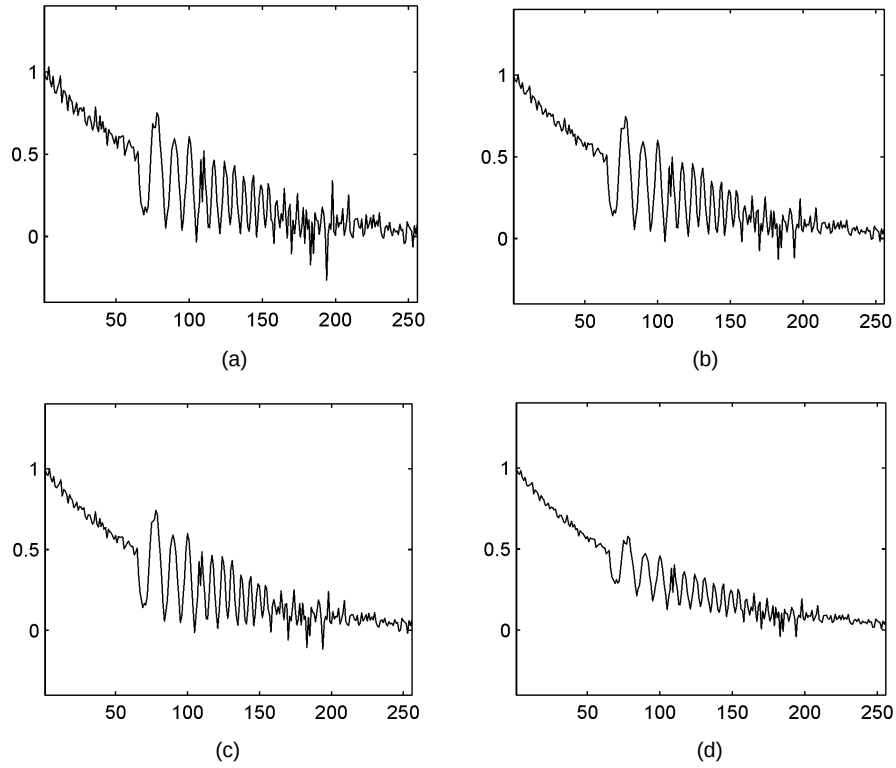


Fig. 9. PMRA of the signal of Fig. 7. (a) NMSE1 = 0.02269. (b) NMSE2 = 0.01373. (c) NMSE3 = 0.012595. (d) NMSE4 = 0.022966.

virtue of Propositions 5 and 8. To take into account the exponential decay of the signal, the center z of the sets $(C_j)_{j\in\mathbb{Z}}$ is chosen as shown in Fig. 8 (z can, for instance, be determined experimentally by fitting an exponential function to the signal x). According to (14), for every $y \in \mathcal{H}_i$, the projections are given by

$$P_j^i y = \begin{cases} z^i + \delta_{i,j} \dfrac{y - z^i}{\left\| y - z^i \right\|_i} & \text{if } \left\| y - z^i \right\|_i > \delta_{i,j} \\ y & \text{otherwise} \end{cases} \tag{30}$$

The multiresolution approximations to x at four scales are displayed in Fig. 9 together with the normalized mean square errors $\text{NMSE}_j = \|x_j - h\|^2 / \|h\|^2$ (Table 1 shows the values of the radii of the balls). It can be observed that the signal obtained at scale $j = 3$ provides the best approximation to the original signal h. The superior ability of the proposed CMRA to capture the salient features of the analyzed signal is illustrated through comparisons with standard LMRAs based on Haar wavelets [14] in Fig. 10 and on third-order spline Battle-Lemarié wavelets [14] in Fig. 11.

4.2 Polyhedral Multiresolution Image Analysis

The $N \times N$ ($N = 128$) 8-bit original image x in Fig. 12 was obtained by observing of a small piece of charcoal through an electron microscope. It will be analyzed via a PMRA in the Hilbert space $\mathcal{H}$ of real $N \times N$ matrices endowed with the Frobenius scalar product: $(\forall (x, y) \in \mathcal{H}^2)\ \langle x \mid y \rangle = \text{trace}(xy^t)$. The PMRA is induced by a smoothness criterion controlled by bounds on the absolute values of the first-order differences in the vertical and horizontal directions (discrete directional derivatives) over a smooth region of the image with support $\mathcal{R} \subset \{1, \ldots, N\}^2$ (the nonsmooth portion of the image was obtained through a simple semiautomatic edge extraction process). Accordingly, the sequence of polyhedra $(C_j)_{j\in\mathbb{Z}}$ is given by

$$(\forall j \in \mathbb{Z})\ \ C_j = C_j^{(v)} \cap C_j^{(h)} \tag{31}$$

with

$$C_j^{(v)} = \left\{ y \in \mathcal{H} \,\middle|\, \max_{\substack{(p,q)\in\mathcal{R} \\ p\neq 1, p\neq N}} \frac{\left| y(p+1,q) - y(p-1,q) \right|}{\sqrt{2}} \le \delta_j \right\} \tag{32}$$

and

$$C_j^{(h)} = \left\{ y \in \mathcal{H} \,\middle|\, \max_{\substack{(p,q)\in\mathcal{R} \\ q\neq 1, q\neq N}} \frac{\left| y(p,q+1) - y(p,q-1) \right|}{\sqrt{2}} \le \delta_j \right\}, \tag{33}$$

where the decreasing sequence $\left(\delta_j\right)_{j\in\mathbb{Z}} \in \mathbb{R}_+^*$ diverges as $j \to -\infty$. These sets are seen to be of the form (17) and, therefore, to yield a PMRA on account of Proposition 7. The approximation x_j to x at resolution j is obtained by alternating rectified projections onto $C_j^{(h)}$ and $C_j^{(v)}$ according to the procedure described in [5] and the associated root mean square approximation error $\text{RMSE}_j = \|x - x_j\| / N$ is computed.

TABLE 1
RADII OF THE BALLS

j	$\delta_{1,j}$	$\delta_{2,j}$	$\delta_{3,j}$
1	0.6333	1.6867	0.8040
2	0.4077	1.6449	0.6436
3	0.3956	1.6241	0.5635
4	0.2799	0.8538	0.4419

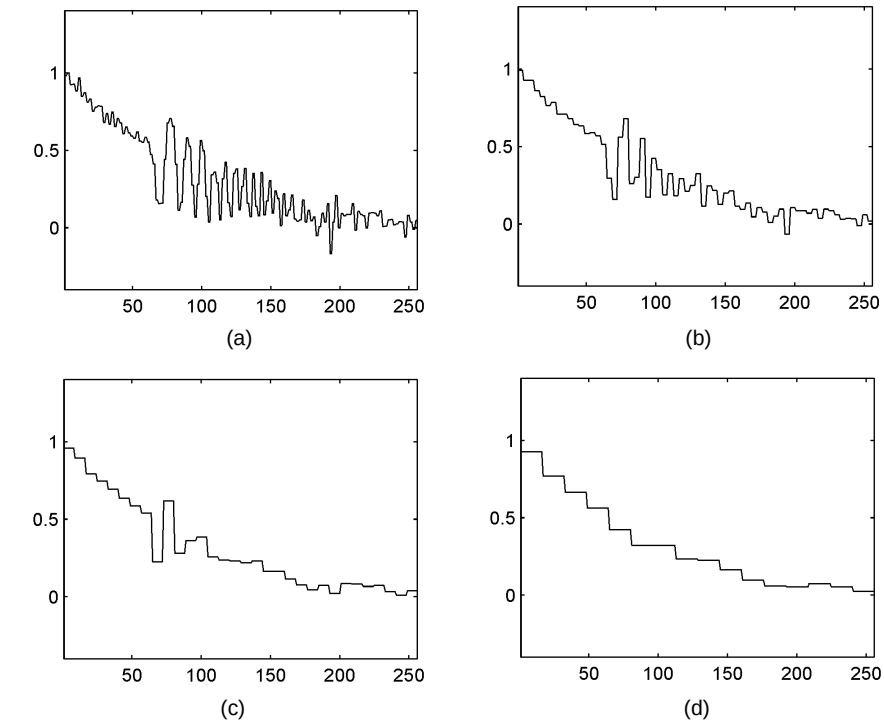


Fig. 10. LMRA of the signal of Fig. 7 based on a Haar wavelet decomposition. (a) NMSE1 = 0.021245. (b) NMSE2 = 0.036598. (c) NMSE3 = 0.047707. (d) NMSE4 = 0.062157.

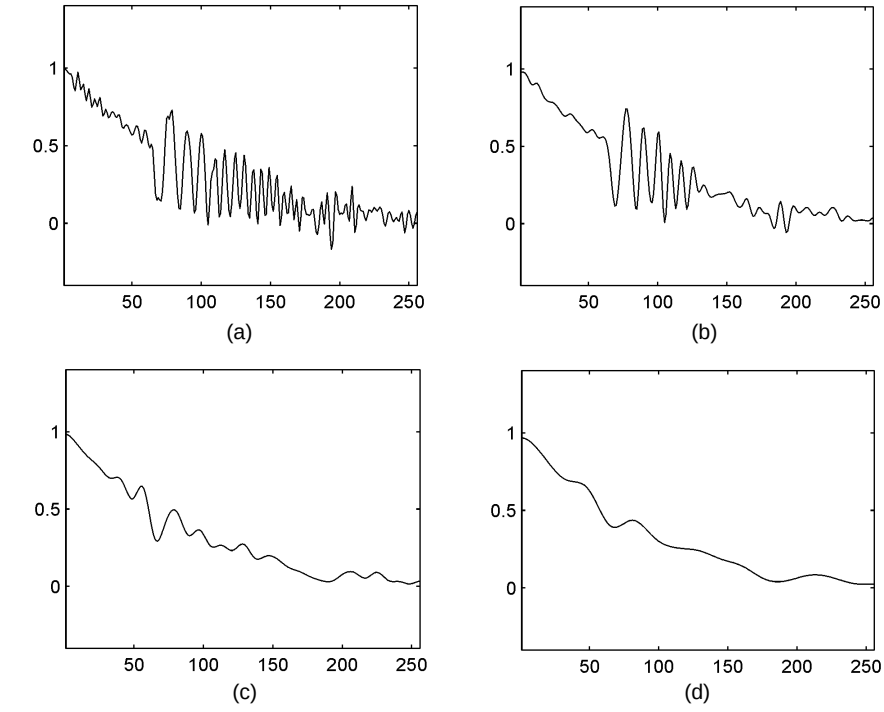


Fig. 11. LMRA of the signal of Fig. 7 based on a third-order spline wavelet decomposition. (a) NMSE1 = 0.014994. (b) NMSE2 = 0.019537. (c) NMSE3 = 0.061574. (d) NMSE4 = 0.06354.

Three approximations produced by the PMRA are depicted for three resolution levels in Fig. 13, Fig. 14, and Fig. 15. These results can be compared to the approximations produced by a standard wavelet LMRA associated with a Haar basis decomposition in Fig. 16, Fig. 17, and Fig. 18, and by a wavelet LMRA associated with third-order spline orthogonal filters [14] in Fig. 19, Fig. 20, and Fig. 21. The parameters $(\delta_j)_{1\leq j\leq 3}$ have been adjusted so as to yield similar RMSE_js for a given j. Unlike the PMRA, the LMRAs generate low-resolution approximations in which the main features of the original image can hardly be recognized.

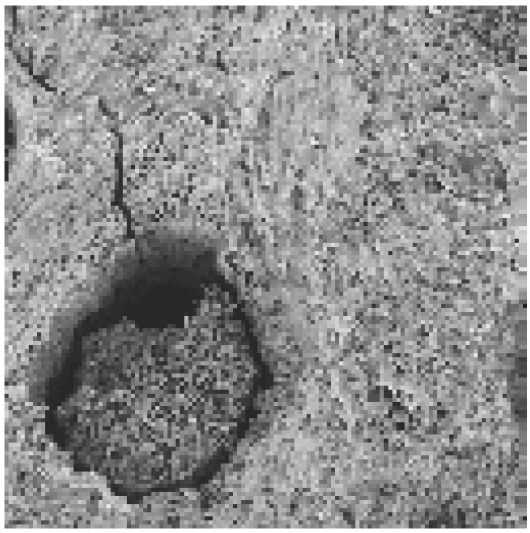


Fig. 12. Original charcoal image.

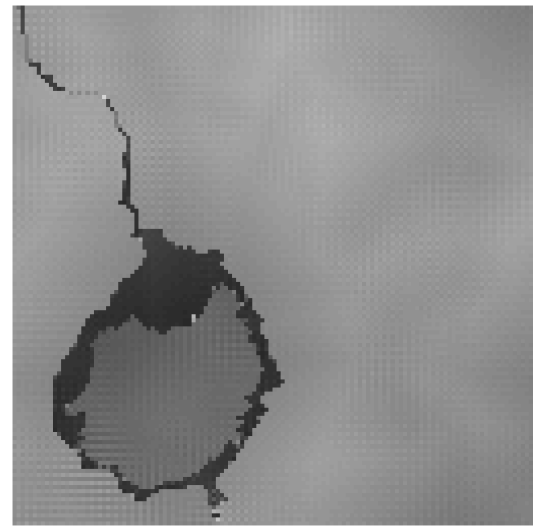


Fig. 15. PMRA – RMSE3 = 0.1225.

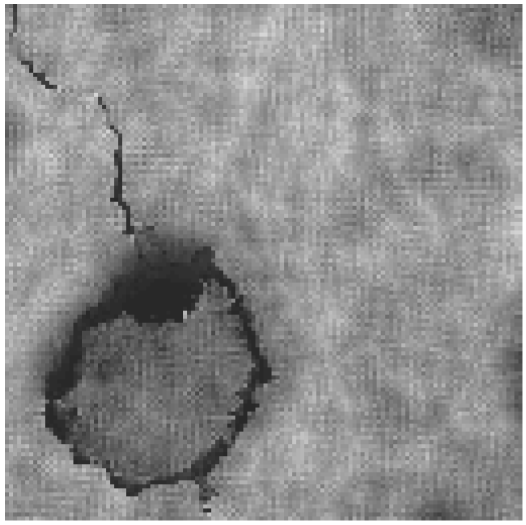


Fig. 13. PMRA – RMSE1 = 0.0938.

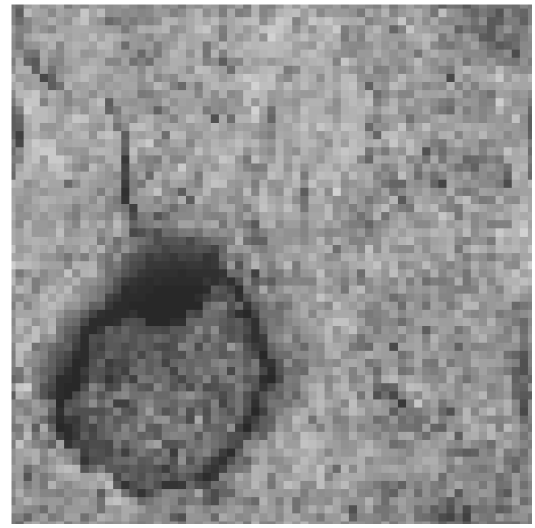


Fig. 16. Haar LMRA – RMSE1 = 0.0970.

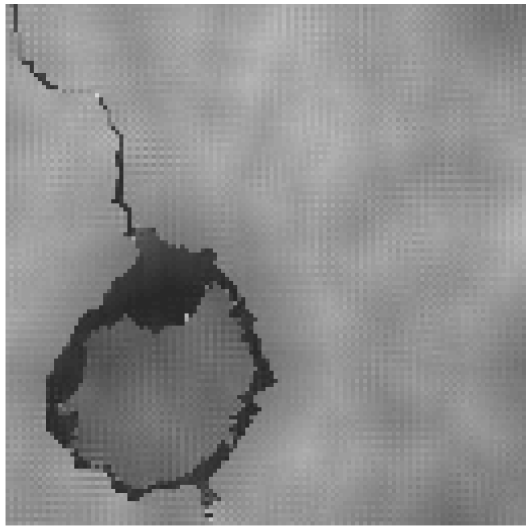


Fig. 14. PMRA – RMSE2 = 0.1158.

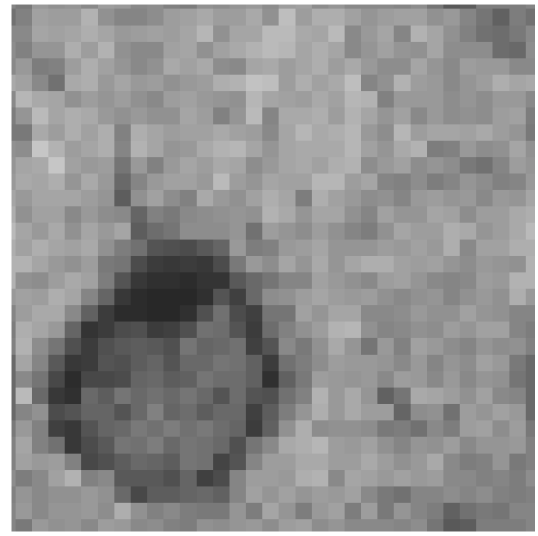


Fig. 17. Haar LMRA – RMSE2 = 0.1171.

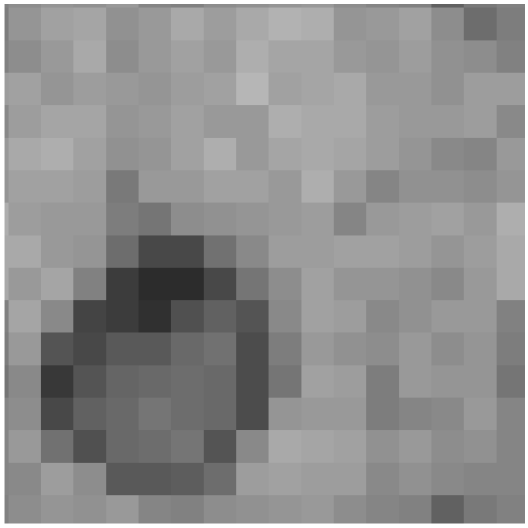


Fig. 18. Haar LMRA – RMSE3 = 0.1264.

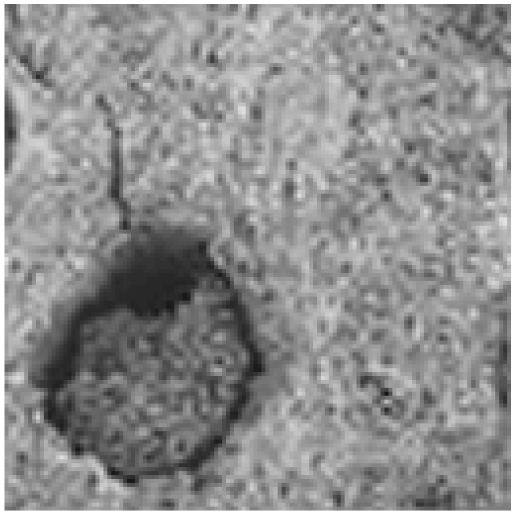


Fig. 19. Spline LMRA – RMSE1 = 0.0931.

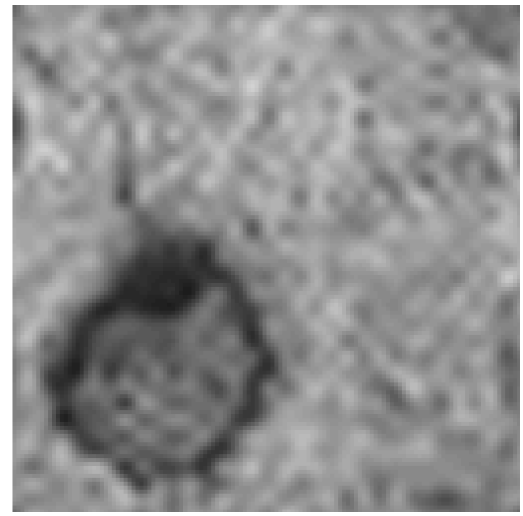


Fig. 20. Spline LMRA – RMSE2 = 0.1131.

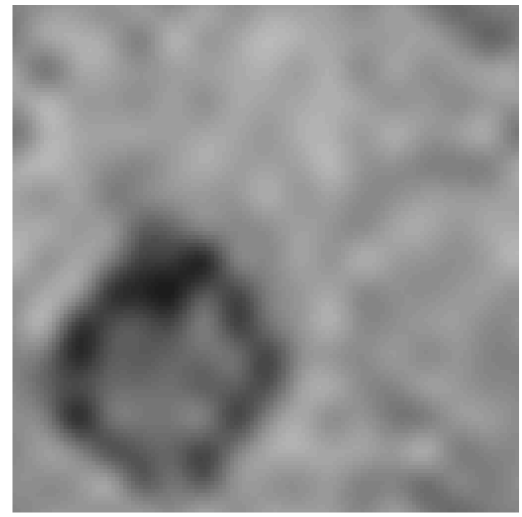


Fig. 21. Spline LMRA – RMSE3 = 0.1239.

5 Conclusion

In this paper, we have presented a geometrical generalization of wavelet multiresolution analysis. The proposed Convex Multiresolution Analysis (CMRA) is defined via projectors onto a sequence of nested closed and convex sets in a Hilbert space. In this recursive formulation, the approximation to a signal at a given scale is obtained by projecting its approximation at the next finer scale onto a convex set. The CMRA framework not only unifies several linear and nonlinear multiresolution techniques but also allows for a wider range of signal features to be captured by the analysis. Several general classes of CMRAs have been investigated and simulation results have been presented to illustrate their potential benefits in signal and image processing problems. As far as computational complexity is concerned, a scale transition in CMRA involves a nonlinear projection, as opposed to a linear one in LMRA. Naturally, the computational cost of a projection depends of the geometrical complexity of the set under consideration. The reader is referred to [12] for examples of simple sets and the closed-form expressions of their projectors. More complex sets can typically be represented as an intersection of simpler ones and decomposition algorithms are available that require only projections onto the simpler sets [11].

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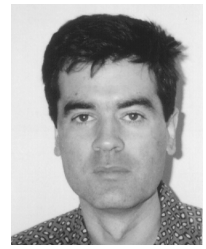
References

[1] J. Babaud, A.P. Witkin, M. Baudin, and R.O. Duda, "Uniqueness of the Gaussian Kernel for Scale-Space Filtering," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 8, no. 1, pp. 26-33, Jan. 1986.

[2] J.A. Bangham, P.D. Ling, and R. Harvey, "Scale-Space From Non-Linear Filters," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 18, pp. 520-528, May 1996.

[3] J.A. Bangham, P. Chardaire, C.J. Pye, and P.D. Ling, "Multiscale Nonlinear Decomposition: The Sieve Decomposition Theorem," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 18, no. 5, pp. 529-539, May 1996.

[4] J. Benedetto and S. Li, "The Theory of Multiresolution Analysis Frames and Applications to Filter Bank." *Applied and Computational Harmonic Analysis*, vol. 5, pp. 389-427, Oct. 1998.
[5] J.P. Boyle and R.L. Dykstra, "A Method For Finding Projections Onto The Intersection Of Convex Sets In Hilbert Spaces," *Lecture Notes in Statistics*, vol. 37, pp. 28-47, 1986.
[6] N. Bourbaki, *Eléments de Mathématique—Espaces Vectoriels Topologiques I-V.* Paris: Masson, 1981. *Elements of Mathematics—Topological Vector Spaces I-V.* New York: Springer-Verlag, 1987.
[7] P.J. Burt and E.H. Adelson, "The Laplacian Pyramid as a Compact Image Code," *IEEE Trans. Comm.*, vol. 31, pp. 532-540, Apr. 1983.
[8] A.E. Çetin and R. Ansari, "Signal Recovery From Wavelet Transform Maxima," *IEEE Trans. Signal Processing*, vol. 42, pp. 194-196, Jan. 1994.
[9] P.G. Ciarlet, *Introduction à l'Analyse Numérique Matricielle et à l'Optimisation.* Paris: Masson, 1988. *Introduction to Numerical Linear Algebra and Optimisation.* New York: Cambridge Univ. Press, 1989.
[10] A. Cohen, "Non-Stationary Multiscale Analysis," *Wavelets—Theory, Algorithms and Applications*, C.K. Chui, L. Montefusco, and L. Puccio, eds., pp. 3-12. San Diego, Calif.: Academic Press, 1994.
[11] P.L. Combettes, "Signal Recovery By Best Feasible Approximation," *IEEE Trans. Image Processing*, vol. 2, pp. 269-271, Apr. 1993.
[12] P.L. Combettes, "The Convex Feasibility Problem in Image Recovery," *Advances in Imaging and Electron Physics*, P. Hawkes, ed., vol. 95, pp. 155-270. New York: Academic Press, 1996.
[13] P.L. Combettes, "Convex Set Theoretic Image Recovery by Extrapolated Iterations of Parallel Subgradient Projections," *IEEE Trans. Image Processing*, vol. 6, pp. 493-506, Apr. 1997.
[14] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, Penn.: SIAM, 1992.
[15] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels.* Paris: Dunod, 1974. *Convex Analysis and Variational Problems.* Amsterdam, The Netherlands: North-Holland, 1976.
[16] F.J. Hampson and J.-C. Pesquet, "*M*-Band Nonlinear Subband Decompositions With Perfect Reconstruction," *IEEE Trans. Image Processing*, vol. 7, pp. 1,547-1,560, Nov. 1998.
[17] S.G. Mallat, "A Theory For Multiresolution Signal Decomposition: The Wavelet Representation," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 11, no. 7, pp. 674-693, July 1989.
[18] S.G. Mallat, "Multiresolution Approximation and Wavelet Orthonormal Bases of $L^2(\mathbb{R})$," *Trans. Am. Math. Soc.*, vol. 315, pp. 69-87, Sept. 1989.
[19] S. Mallat and S. Zhong, "Characterization of signals from multiscale edges," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 14, pp. 710-732, July 1992.
[20] Y. Meyer, *Ondelettes et Opérateurs.* Paris: Hermann, 1990.
[21] J.-M. Morel and S. Solimini, *Variational Methods in Image Segmentation*. Boston, Mass.: Birkhäuser, 1995.
[22] P.Y. Simard and G.E. Mailloux, "A Projection Operator For The Restoration of Divergence-Free Vector Fields," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 10, no. 2, pp. 248-256, Mar. 1988.
[23] Y. I. Wong, "Nonlinear Scale-Space Filtering and Multiresolution System," *IEEE Trans. Image Processing*, vol. 4, pp. 774-787, June 1995.



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