Signal Recovery by Best Feasible Approximation

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Abstract— **The objective of set theoretic signal recovery is to find a feasible signal in the form of a point in the intersection S of sets modeling the information available about the problem. For problems in which the true signal is known to lie near a reference signal r, the solution should not be any feasible point but one which best approximates r, i.e., a projection of r onto S. Such a solution cannot be obtained by the feasibility algorithms currently in use, e.g., the method of projections onto convex sets (POCS) and its offsprings. In this work, methods to project a point onto the intersection of closed and convex sets in a Hilbert space are introduced and applied to signal recovery by best feasible approximation of a reference signal. These recent algorithms are closely related to the above projection methods, to which they add little computational complexity.**

I. Introduction

The set theoretic signal recovery (restoration or reconstruction) problem is to produce an estimate of an original signal which is consistent with all prior knowledge as well as with the data. The set of signals consistent with a particular piece of information is called a property set. The intersection of these property sets is the set of feasible signals, i.e., the solution set. The readers unfamiliar with set theoretic estimation are referred to [2].

In many recovery problems, the original signal is known to lie in the neighborhood of a reference signal r, e.g., [10], [13]. Such a constraint is of great interest as it can significantly reduce the set of feasible solutions and thereby improve the quality of the set theoretic estimates. If the solution space is a Hilbert space $(\Xi, \|\cdot\|)$, the property set associated with this information is the ball $B = \{a \in \Xi \mid \|a - r\| \leq \delta\}$. Let $(S_i)_{1 \leq i \leq m}$ be the property sets based on the rest of the information and S their intersection (we restrict our attention to cases when the S_i's are closed and convex). The recovery problem is then to find a point in $S' = S_1 \cap \cdots \cap S_m \cap B = S \cap B$. If δ were known exactly, a point in S' could be obtained via the standard projection algorithms. However, in practice, δ must generally be determined heuristically and B cannot be constructed reliably. The problem is then to find a point in S', given that B is an unknown ball centered at r and intersecting with S. The following proposition indicates that such a solution can be obtained by taking the point a^* in S which lies closest to r, i.e., the best approximation of r by a feasible point. Geometrically, a^* is simply the projection of r onto S (see Fig. 1).

Proposition 1: Let a^* be the projection of r onto S. Then $a^* \in S'$.

Let us stress that, in accordance with the principles of set theoretic estimation, all the points in S' are equally valid solutions [2]. However, δ being unknown, a^* is the only point that can be guaranteed to lie in S' and it must therefore be selected as the solution. The set theoretic estimation problem now becomes a best approximation problem that can be states as

Find

$$a^* \in S = \bigcap_{i=1}^{m} S_i$$

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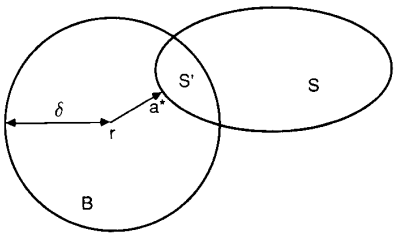


Fig. 1. Best feasible approximation.

such that

$$(\forall\, b \in S) \quad \|a^* - r\| \leq \|b - r\|. \tag{1}$$

The algorithms currently used to synthesize set theoretic estimates guarantee only a feasible solution, not an actual projection onto the feasibility set. In this correspondence, we introduce two recently developed projection algorithms that converge to the projection of the starting point onto the intersection of closed and convex subsets of a Hilbert space.

Our notations and assumptions will be as follows. The true signal h belongs to a Hilbert space Ξ with norm $\|\cdot\|$ and scalar product $\langle \cdot \mid \cdot \rangle$. Convergence in Ξ is understood to be strong convergence. The finite-dimensional Hilbert space of real N-tuples equipped with the scalar product $\langle a \mid b \rangle = a^t Q b$, where Q is a $N \times N$ positive definite symmetric matrix, is denoted by $\mathbb{H}^N$. In applied work, in order to be amenable to digital processing, problems are eventually reformulated in $\mathbb{H}^N$ via discretization or parametrization. The information available about the recovery problem is associated with m closed and convex property sets in Ξ denoted by $(S_i)_{1 \leq i \leq m}$. The pair $\left(\Xi, (S_i)_{1 \leq i \leq m}\right)$ is called a set theoretic formulation and $S = \bigcap_{i=1}^{m} S_i$ the feasibility (or solution) set. Unless otherwise stated, the set theoretic formulation is consistent, i.e., $S \neq \varnothing$. The projection operator onto S_i is denoted by P_i. Finally, a^* designates the solution to (1), i.e., the projection of r onto S.

II. Feasibility Algorithms

Given a set theoretic formulation $(\Xi, (S_i)_{1 \leq i \leq m})$ the set theoretic recovery problem is to find a point a in S. In signal recovery, methods which employ projections onto the individual sets have proven quite useful to solve this feasibility problem. The most popular scheme, known as POCS for projection onto convex sets, consists of projecting an initial estimate a_0 onto each set in a cyclic manner as described by the following algorithm:

$$\begin{aligned} &c_0^m = a_0 \\ &\text{for } n = 1, 2, \cdots \\ &\quad \left| \begin{aligned} &c_n^0 = c_{n-1}^m \\ &\text{for } i = 1, \cdots, m \\ &\quad \left\lfloor c_{n+1}^i = P_i\left(c_n^{i-1}\right) \right. \\ &a_n = c_n^m. \end{aligned} \right. \end{aligned} \tag{2}$$

In POCS, only one set is utilized at each elementary step. An alternative procedure is to activate all the sets simultaneously at each

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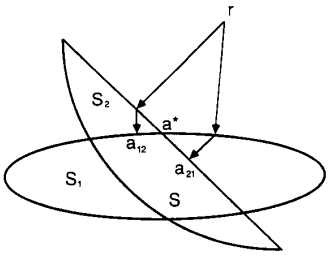


Fig. 2. Inadequacy of POCS.

step. In such a scheme, the new iterate is a convex combination of the projections of the current iterate onto each set. Given an initial estimate a_0, the corresponding algorithm is as follows:

$$\begin{array}{l}\text{for } i = 1, \cdots, m \\ \quad \lfloor c_1^i = a_0 \\ \text{for } n = 1, 2, \cdots \\ \quad \left| \begin{array}{l} a_n = \Sigma_{i=1}^m w_i P_i\left(c_n^i\right) \\ \text{for } i = 1, \cdots, m \\ \quad \lfloor c_{n+1}^i = a_n \end{array} \right.\end{array} \tag{3}$$

where the weight satisfy

$$\sum_{i=1}^m w_i = 1 \quad \text{and} \quad (\forall\, i \in \{1, \cdots, m\}) \quad w_i > 0. \tag{4}$$

Proofs of the convergence of (2) and (3) have appeared in several places (see [2] for details).

In [14], experimental evidence showed that by letting the initial estimate in POCS be a signal with desired properties, one could insert additional *a priori* information and qualitatively affect the solution. This suggests that POCS converges to a point close to the initial estimate and one is naturally led to ask whether by using the reference signal r as an initial estimate (1) can be solved by the above feasibility algorithms. If the S_i's are affine subspaces, i.e., translates of vector subspaces, the following proposition gives an affirmative answer.

Proposition 2: Let $(S_i)_{1 \le i \le m}$ be a collection of intersecting closed affine subspaces in a Hilbert space Ξ and let $a_0 = r$. Then $(a_n)_{n \ge 0}$ in (2) and (3) converges to a^*.

Proof: For vector subspaces, the convergence of the cyclic algorithm (2) was established by Halperin [5]. The extension to affine subspaces follows readily. The convergence of the parallel algorithm (3) is proved in [12].

Unfortunately, the conclusion of Proposition 2 cannot be generalized to arbitrary convex sets. In this respect, Fig. 2 provides a simple counterexample in the Euclidean plane where, regardless of the order of the projections, POCS fails to converge to the projection a^* of r onto S. Alternative methods to solve (1) have been proposed in the literature but they do not apply to our general conditions. For instance, the method proposed in [7] is limited to the case when all the S_i's are half-spaces in finite dimension. On the other hand, the method developed in [11] is limited to the case $m = 2$ and is relatively involved.

III. Algorithms for Best Feasible Approximation

Consider the following cyclic algorithm.

$$\begin{array}{l} a_0^m = a_0 \\ \text{for } i = 1, \cdots, m \\ \quad \lfloor b_0^i = 0 \\ \text{for } n = 1, 2, \cdots \\ \quad \left| \begin{array}{l} a_n^0 = a_{n-1}^m \\ \text{for } i = 1, \cdots, m \\ \quad \left| \begin{array}{l} c_n^i = a_n^{i-1} + b_{n-1}^i \\ a_n^i = P_i\left(c_n^i\right) \\ b_n^i = c_n^i - P_i\left(c_n^i\right) \end{array} \right. \\ a_n = a_n^m \end{array} \right.\end{array} \tag{5}$$

Proposition 3: Let $a_0 = r$. Then $(a_n)_{n \ge 0}$ in (5) converges to a^*.

Proof: For the case $\Xi = \mathbb{H}^N$, the proof is given in [3] for convex cones and in [6] for convex sets whose intersection has a nonempty interior. Convergence in the general case of arbitrary closed and convex sets in an arbitrary Hilbert space is established in [1] and [4].

Algorithm (5) differs from (2) in that the component b_{n-1}^i is added to a_n^{i-1} before performing the projection onto S_i. This component is the outward pointing normal of the set S_i at the projection, a_{n-1}^i, of the previous iterate, c_{n-1}^i, onto S_i. Now suppose that the S_i's are affine subspaces, say $S_i = \{a + d_i \mid a \in \vec{S}_i\}$, where $\vec{S}_i$ is a vector subspace. Then, the linearity of $\vec{P}_i$, the projection operator onto $\vec{S}_i$, yields

$$\begin{aligned} a_n^i &= P_i\left(a_n^{i-1} + b_{n-1}^i\right) \\ &= d_i + \vec{P}_i\left(a_n^{i-1} + b_{n-1}^i\right) \\ &= d_i + \vec{P}_i\left(a_n^{i-1}\right) + \vec{P}_i\left(b_{n-1}^i\right). \end{aligned} \tag{6}$$

But, since $b_{n-1}^i \perp S_i$, we get $\vec{P}_i\left(b_{n-1}^i\right) = 0$. Hence

$$a_n^i = d_i + \vec{P}_i\left(a_n^{i-1}\right) = P_i\left(a_n^{i-1}\right). \tag{7}$$

It follows that (5) is equivalent to (2) and, as expected, Proposition 3 coincides with Proposition 2. In general, (7) indicates that b_{n-1}^i need not be computed if S_i is an affine subspace.

The recursion (5) can be modified by activating the property sets simultaneously, as opposed to cyclically, at each iteration. This leads to the algorithm:

$$\begin{array}{l}\text{for } i = 1, \cdots, m \\ \quad \lfloor c_1^i = a_0 \\ \text{for } n = 1, 2, \cdots \\ \quad \left| \begin{array}{l} a_n = \sum_{i=1}^m w_i P_i\left(c_n^i\right) \\ \text{for } i = 1, \cdots, m \\ \quad \left| \begin{array}{l} b_n^i = c_n^i - P_i\left(c_n^i\right) \\ c_{n+1}^i = a_n + b_n^i \end{array} \right. \end{array} \right.\end{array} \tag{8}$$

where the weights satisfy (4).

Proposition 4: [8] Let $\Xi = \mathbb{H}^N$ and $a_0 = r$. Then $(a_n)_{n \ge 0}$ in (8) converges to a^*.

Again, we note that this recursion is similar to the parallel scheme (3), except for the fact that the current iterate is modified by adding the outer normal component obtained at the previous iteration before projecting.

IV. Inconsistent Formulations

All the results presented above rest on the assumption $S \neq \emptyset$. In some problems, however, the set theoretic formulation may be inconsistent as a result of two or more incompatible constraints [2]. The properties of (2) and (3) in this case are discussed in [2] and we shall now discuss those of (5) and (8).

Let $\Xi = \mathbb{H}^N$ and let $(\overline{a}_i)_{1\leq i\leq m}$ be stationary points, i.e., $P_1(\overline{a}_m) = \overline{a}_1$ and, for every i in $\{2,\cdots,m\}$, $P_i(\overline{a}_{i-1}) = \overline{a}_i$. Note that $\overline{a}_i$ is guaranteed to lie only in S_i and could be quite distant from the other sets, depending on the severity of the inconsistency. It was shown in [8] that, if $m = 2$, (5) converges to the stationary point $\overline{a}_m$ in S_m which lies closest to the initial point a_0. However, this result could not be extended to arbitrary m without overly restrictive and impractical assumptions. In fact, even for affine subspaces, convergence may fail [9]. Therefore, (5) should not be used with inconsistent set theoretic formulations.

To study the behavior of (8), it is convenient to introduce the convex functional

$$\begin{aligned}\Phi : \Xi &\to [0,+\infty[\\ a &\mapsto \sum_{i=1}^{m} w_i \|a - P_i(a)\|^2 \end{aligned} \tag{9}$$

and the set G of points in Ξ which minimize Φ. It is easy to see that in the inconsistent case $G = S$ since the feasibility set is the set of solutions of the equation $\Phi(a) = 0$. In the consistent case, this equation has no solution but one can look for an approximate one by replacing (1) by the more general problem

Find

$$\tilde{a} \in G$$

such that

$$(\forall\, b \in G) \quad \|\tilde{a} - r\| \leq \|b - r\|. \tag{10}$$

Proposition 5: [8] Let $\Xi = \mathbb{H}^N$, $a_0 = r$, and $S = \emptyset$. Then $G \neq \emptyset$ and $(a_n)_{n>0}$ in (8) converges to the projection $\tilde{a}$ of r onto G if any of the following conditions holds:

i) all the S_i's are polyhedra;

ii) $\left(\exists (b_i^*)_{1\leq i\leq m} \in \mathrm{X} S_i\right)\ (\forall\, (i,j) \in \{1,\cdots,m\}^2)$ $\|b_i^* - b_j^*\| = \inf\{\|b_i - b_j\| \,|\, b_i \in S_i, b_j \in S_j\}$;

iii) one of the S_i's is bounded.

In concrete applications (iii) is satisfied since there are always bounds on the amplitude of the original signal. The practical implications of Proposition 5 are of great importance: (8) yields the (global) minimizer of the weighted average of the squares of the distances to the property sets which is the closest to the reference signal. In that sense, such a solution is the one which is closest to r and to satisfying all the constraints. Therefore, for those best feasible approximation problems in which the consistency of the set theoretic formulation is questionable, one should employ the parallel algorithm (8) instead of its serial counterpart (5).

V. Discussion

In this paper, we have considered the problem of recovering a signal by best feasible approximation of a reference signal. Two simple algorithms have been proposed to solve this problem. The first algorithm (5) activates the set in a cyclic fashion while the second (8) activates them simultaneously. They are slightly more complex than the feasibility algorithms (2) and (3) but, just like then, require only the ability to compute projections.

In our applications, we have found that the rate of convergence of (8) is affected by the choice of the weights $(w_i)_{1\leq i\leq m}$ and is usually slower than that of (5). On the other hand, a significant advantage of the parallel scheme over its serial counterpart is to be easily implementable on concurrent processors since the projections and the outer normal can be evaluated simultaneously at each iteration. Overall, these factors favor using the parallel algorithm (8) if a large parallel architecture is available and the serial one otherwise. Moreover, if the set theoretic formulation is inconsistent, a significant advantage of (8) is to converge to a weighted least-squares solution.

In certain recovery problems it has been argued that the feasible solution of minimum energy was the most appropriate estimate and rather complex procedures have been proposed to obtain such a solution in particular cases [11]. One who adopts this aim can obtain a minimum energy feasible solution in the general convex case treated in this paper by simply letting r be the zero signal.

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