are general and can be extended to the case of more detailed machine modeling.

The steady state obtained by the transient control will in general not be the steady state desired by the system operators. However, once the sustained oscillation has been quenched, the system operators can easily adjust the steady-state operating point. In other words, once the oscillation has been removed, existing system control procedures can be used in their normal routine fashion. Excitation control, if present, could potentially conflict with the proposed governor/turbine control, if the chosen feedback gains reduce the time constant of the governor/turbine to the same order of magnitude as that of the excitation system. The conflict can be avoided by limiting the feedback gain for the governor/turbine so that its time constant is large enough not to conflict with the excitation control. The size of the time constant will have to be determined by simulation. However, the loss of gain on the governor/turbine side may be more than made up by the presence of excitation control.

In this paper we have chosen to use variations in the mechanical output of the machines for the control. We could very well have used control of the excitation system instead. This has appeal, because excitation control will be much faster. The main question is whether excitation control is strong enough to control the system during the transient period following a major disturbance. That is an issue that will be addressed in the future. It may be that the local feedback control will have to include both excitation and governor/turbine control. Our main goal in this paper was to set forth the analysis of the control strategy in the simplest setting. More complex machine models can be analyzed, but the algebraic difficulties increase rapidly.

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# A Bound for the Zeros of Polynomials


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*Abstract*—**In this paper we introduce a new bound for the zeros of polynomials. Our result is obtained by applying Cauchy's theorem to a polynomial derived from the original polynomial through an exponential transform.**


## I. Introduction

The problem of determining bounds on the magnitude of the zeros of a polynomial has a long history that dates back to the work of Cauchy [2], who gave a very simple expression for the bound in terms of the coefficients of the polynomial. An account of several developments on this topic can be found in the comprehensive book by Marden [5]. Recently, Zeheb [6] gave new extensions of Cauchy's result, by introducing polynomial transformations, treating separately cases when the original polynomial is real and complex. Minimizing Cauchy's bound for a transformed pair first, Zeheb defined two circular bounds for the original polynomial. His result demonstrated improvement over known results.

Starting from the idea of determining a bound on the zeros of polynomial, determining Cauchy's bound on zeros of its transformed pair first, we will define a new circular bound. Our transformation is based on a nonlinear transformation of the variable, which conceptually should give us a better upper bound. The advantage of such a transformation will be illustrated through several examples, where our results are compared to the existing ones. This paper will show the improvement over the existing bounds, when they are applied to the transformed polynomial. It will also examine convergence of the bound after iterative transformations of the original polynomial.

### A. Bound for the Zeros of a Polynomial

Let

$$P(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \qquad a_k \in \mathbb{C} \tag{1}$$

be a complex polynomial of degree $n$.

*Theorem:* All zeros of $P$ lie in the disk

$$\{ z \in \mathbb{C} \mid |z| < \sqrt{1 + A} \} \tag{2}$$

with

$$A = \max_{0 \le k \le n-1} \{ | a_k^2 + 2(-1)^k (B - C) | \} \tag{3}$$

where

$$B = \sum_{\substack{0 \le i < j \le [n/2] \\ i+j=k}} a_{2i} a_{2j} \tag{4}$$

$$C = \sum_{\substack{0 \le i < j \le [(n-1)/2] \\ i+j=k-1}} a_{2i+1} a_{2j+1} \tag{5}$$

$a_n = 1$, and $[l]$ denotes the integer part of $l$.


Manuscript received May 6, 1991. This paper was recommended by Associate Editor P. Agathoklis.



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IEEE Log Number 9201115.



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*Proof:* Let us express $P$ as a sum of even and odd degree polynomials $P = P_e + P_o$. If $P(z) = 0$, then $P_e^2(z) - P_o^2(z) = 0$ as well. It follows from (1) that

$$P_e^2(z) - P_o^2(z) = \sum_{k=0}^{n} \{(-1)^k a_k^2 + 2(B - C)\} z^{2k}. \quad (6)$$

Let us define a polynomial

$$Q(z) = (-1)^n (P_e^2(z) - P_o^2(z)) = \sum_{k=0}^{n} a_k^{(1)} z^{2k} \quad (7)$$

with

$$a_k^{(1)} = (-1)^n \{(-1)^k a_k^2 + 2(B - C)\}. \quad (8)$$

The change of variable $z_1 = z^2$ in $Q$ leads to the new polynomial

$$P_1(z_1) = \sum_{k=0}^{n} a_k^{(1)} z_1^k. \quad (9)$$

From the expression for $a_k$ it can be seen that $a_n^{(1)} = 1$ and Cauchy's bound can be applied to $P_1$, which gives the following bound for the zeros of $P_1$

$$|z_1| < 1 + A \quad (10)$$

where $A = \max_{0 \le k \le n-1} \{|a_k^{(1)}|\}$. We conclude that a bound for the zeros of $P$ is

$$|z| < \sqrt{1 + A}. \quad (11)$$

*Example 1:* Let

$$P(z) = z^3 + 3z^2 + 2z + 1. \quad (12)$$

Using (3), $A = \max\{1, 2, 5\}$ and the bound on the zeros of $P(z)$ in (12) is

$$|z| < \sqrt{6} \approx 2.45. \quad (13)$$

This example was given in [6], where Zeheb's bound is $|z| < 2.75$ [6, theorem 1], showing the improvement over existing bounds given by Cauchy [2] $|z| < 4$, Marden [5, theorem 27.3] $|z| \le 3.627$, [5, equation (27.19)] $|z| < 3.873$, [5, equation (27.25)] $|z| \le 2.828$, [5, equation (27.26)] $|z| \le 5.414$, Joyal *et al.* [4, theorem 1] $|z| \le 4$, [4, theorem 1 (corollary 2)] $|z| \le 3.193$, [4, theorem 1 (corollary 3)] $|z| \le 3$, Datt and Govil [3, theorem 1] $|z| \le 3.951$, [3, theorem 2)] $|z| \le 3.953$, Boese and Luther [1] $|z| \le 3.951$.

## II. Bounds Improvement

If the bound (2) is applied to the polynomial $P_1$ in (9) instead of $P$ in (1), looking for $A$ in terms of the $a_k^{(1)}$'s the bound for all zeros of $P$ can be further improved, bearing in mind that $|z| = \sqrt{|z_1|}$. All other existing bounds can also be improved when applied to $P_1$. This is illustrated in the following examples.

*Example 2:* Let

$$P(z) = z^3 + 3z^2 + 2z + 1 \quad (14)$$

as in Example 1. Using (8) and (9),

$$P_1(z_1) = z_1^3 + 5z_1^2 - 2z_1 - 1. \quad (15)$$

From (3) and (2), $|z_1| < \sqrt{30}$ and finally

$$|z| < \sqrt{\sqrt{30}} \approx 2.34 < 2.45. \quad (16)$$

*Example 3:* Let

$$P(z) = z^3 + 0.1z^2 + 0.1z + 0.04. \quad (17)$$

According to [6, theorem 2], where the same example was also given, we get $|z| < 0.65$. Now, starting from $P$, using (8) and (9), we obtain

$$P_1(z_1) = z_1^3 + 0.19z_1^2 + 0.002z_1 - 0.0016. \quad (18)$$

Referring to the same theorem and using the fact that $|z| = \sqrt{|z_1|}$, we find that all the zeros of $P(z)$ lie in the disk $\{z \in \mathbb{C} \mid |z| < 0.4382\}$.

*Example 4:* Let

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1. \quad (19)$$

According to (8) and (9),

$$P_1(z_1) = z_1^4 - 10z_1^3 - 5z_1^2 + 2z_1 + 1. \quad (20)$$

Referring to [3, theorem 2] we get $|z_1| \le 10.999$. Hence, $|z| \le \sqrt{10.999} \approx 3.316$, which is better then the original bound $|z| \le 4.99$ for $P$ in (19), obtained according to [3, theorem 2].

## III. Recursive Application of the Bound

As can be seen from (16), the bound (2) on the zeros of the original polynomial $P$ in (1) is improved when applied to the transformed polynomial $P_1$ in (9). This procedure can be applied iteratively. $P_1$ in (9) can be further transformed using (8) into $P_2$ and $P_2$ into $P_3$ until the last iteration brings no practical improvement. In practice it turns out that very few iterations are needed. If the modulus of one of the polynomial coefficients is much greater than the others, the bound for all zeros of the polynomial $P_r(z_r)$ is approximately equal to $\sqrt{1 + |a_k^{(r)}|_{\max}^2}$, meaning that the iterative procedure is over when $|a_k^{(r+1)}|_{\max} \approx |a_k^{(r)}|_{\max}^2$. Then all zeros of the original polynomial $P(z)$, lie in the disk

$$\left\{z \in \mathbb{C} \mid |z| < \left(1 + |a_k^{(r)}|_{\max}\right)^{2^{-r}}\right\}. \quad (21)$$

If we start the iterative procedure on $P(z)$ given in Example 1, then

$$\begin{aligned} P_1(z_1) &= z_1^3 - 5z_1^2 - 2z_1 - 1 \\ P_2(z_2) &= z_2^3 - 29z_2^2 - 6z_2 - 1 \\ P_3(z_3) &= z_3^3 - 853z_3^2 - 22z_3 - 1 \\ P_4(z_4) &= z_4^3 - 727653z_4^2 - 1222z_4 - 1. \end{aligned} \quad (22)$$

Obviously $|a_2^{(4)}| \approx |a_2^{(3)}|^2$, hence $|z| < (1 + 853)^{1/8} \approx 2.325$. This additional improvement brings the bound very close to the exact one (numerically determined as $|z| \le 2.3247$). However, the iterative procedure does not necessarily lead to the exact bound. Still, that problem can be solved with the help of [6, theorem 2] as shown in the following example.

*Example 5:* Let

$$P(z) = z^2 - 5z + 6. \quad (23)$$

The exact bound $|z| \le 3$ can be easily found by solving the quadratic equation. On the other hand, all bounds [1]–[6] are very far from the exact one, being in the range from 5.288 [6, theorem 1] to 13.93 [5, equation (27.25)]. After just two iterations we have

$$P_2(z_2) = z_2^2 - 97z_2 + 1296. \quad (24)$$

From this it can be seen that bound converges to $|z| < 6$. But [6,

theorem 2] leads to $|z_2| \leq 97 + 1296/97 = 110.36$. Hence, $|z| \leq (110.36)^{1/4} \approx 3.24$, which is very close to the exact solution. This result can be further improved if we proceed iteratively and then apply [6, theorem 2], converging to the exact bound.

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## Varistor Control of Inductive Transients

Joseph M. Diamond

*Abstract*—**It is shown that the varistor, a nonlinear resistor with current-voltage characteristics of the form $i = ke^n$, can be used to achieve superior limitation of inductive transients over that obtainable with a simple diode, or with a diode plus additional linear resistance. To this end a transient analysis is made of the circuit composed of the inductor, its internal linear resistance, and a shunt varistor when the source circuit is opened. The resultant differential equation, though nonlinear, is integrable in terms of elementary functions, and general results are presented in mathematical and graphical form for the current and voltage transients. The results span the spectrum from the linear resistor ($n = 1$) to the zener diode ($n = \infty$), with intermediate values representative of silicon carbide, selenium, and zinc oxide devices. For all $n > 1$, the transient is found to reach zero in a finite time rather than asymptotically. For a given voltage rise when the source circuit is opened (a basic design parameter), the zener diode yields the shortest transient. Except for the lowest values of $n$, the varistor is shown to approach the performance of the zener, while offering practical advantages.**

### I. General

When an inductive circuit carrying current is interrupted, the current cannot change instantaneously—an alternate path for the current must be provided. For the relay coil or other inductive device of Fig. 1, for example, a shunt path (diode $D$ plus resistor $Z$, which may be linear or nonlinear) is pr ovided to carry the coil current when switch $S$ is opened. Without this path, the current will have to flow into whatever stray capacitance is present, produ cing a voltage spike which, if not damped by some form of breakdown, can easily reach thousands of volts.[1] The spike may arc across the switch contac ts, damage the coil insulation or damage the switch when the latter is a semico nductor [1, pp. 11, 12, 16, 17].

Manuscript received May 14, 1991; revised February 26, 1992. This paper was recommended by Associate Editor C. C. Liu.

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IEEE Log Number 9201314.

[1] If $C$ is the stray capacitance, and the resonant $LC$ circuit is undamped, the peak value of the oscillatory current will equal the initial current $I_o$. The peak voltage, reached 1/4 cycle after the peak current, will be $I_o \omega L = I_o(1/\sqrt{LC})L = I_o\sqrt{L/C}$. For an idea of the magnitudes involved, consider $I_o = 50$ mA, $L = 2$ H, $C = 200$ pF; we find $I_o\sqrt{L/C} = 5000$ V.

### II. Diode Protection

A common form of protection is the well-named "free wheeling diode," which is simply the diode $D$ of Fig. 1 with $Z = 0$. When the switch is opened, the coil current flows around the closed path shown until it dies away in accordance with the time constant $L/r$, slightly modified by the diode drop. While simple and effective, the method has a disadvantage: the resultant time constant may be undesirably long. For example, when the inductance represents a relay coil, the relay's own contacts may open rather slowly, prolonging the arc when they carry substantial currents. The solution is to add resistance at $Z$ to reduce the time constant. The voltage $e$ will increase, but that is inherent—it is this voltage that opposes the coil current $i$, and the higher the permissible value of $e$, the faster the current will decline. This factor is basic for all forms of protection, and in the analysis of the varistor circuit below, the ratio of the initial transient voltage $E_o$ to the operating voltage $E$ is a basic design parameter, represented by the symbol $a$.

### III. Varistor Protection

Fortunately an avenue is available for limiting the duration of the transient for a given permissible level of $e$, namely, the waveform of $e$. Clearly, if $e$ can persist near its high initial value longer, it will depress the coil current more strongly. The ultimate in this direction is the use of a constant voltage device such as a zener diode (the diode $D$ will still be needed to prevent conduction while the coil is energized). Least advantageous is the use of a linear resistor for $Z$—in this case $e$ and $i$ decline together exponentially.

Intermediate in character between the linear resistor and the zener diode are the nonlinear resistors known as varistors. They include the silicon carbide, selenium and zinc oxide (MOV) varieties. All follow a law of the form:

$$i = ke^n. \tag{1}$$

Representative (but not limiting) values for $n$ are 2 to 4 for silicon carbide, 8 for selenium, and 16 for zinc oxide [1, pp. 34-37]. For the linear resistor $n = 1$, while as $n$ approaches infinity, we approach the constant voltage device. Thus (1) covers the spectrum from the linear resistor to the zener diode as $n$ varies from 1 to infinity. Varistors are compact, reliable, inexpensive, and, unlike many zeners, capable of considerable energy absorption.[2] Their use in controlling inductive transients is well known [1, pp. 73-79], but a general analysis of the transient response of the inductor/varistor circuit has been lacking. As will be seen, the performance of varistors, except for the lowest values of $n$, approaches that of the zener as far as shortening the transient is concerned.

### IV. Transient Analysis of the Varistor Circuit

If $I_o$ is the coil current with $S$ closed, we have:

$$I_o = E/r. \tag{2}$$

When $S$ is opened, $I_o$ will be the initial value of the transient current $i$. If $E_o$ is the initial value of the transient voltage $e$, we have, neglecting the small voltage drop in $D$:

$$I_o = kE_o^n. \tag{3}$$

[2] There is, however, a class of zener diodes designed for energy absorption (trade name "TransZorbs") [2, section 1].

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